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Automated Support for Battle Operational-Strategic Decision-Making

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Abstract

Armies have always felt the need to base their decisions on proven operational research methods that seek to provide the command with alternatives in the decision-making process, from optimization of operations to strategic evaluation and cost economics.
Battle casualties are a subject of study in military operations research, which applies mathematical models to quantify the probability of victory *vs.* loss. In particular, different approaches have been proposed to model the course of battles. However, none of them provide adequate decision-making support for high-level command. To overcome this situation, this thesis proposes an innovative framework that overcomes most limitations of traditional models and supports decision-making at the highest command levels: the strategic and the operational ones, resorting to the determination of the decay of combat force levels, commonly referred to as *attrition* (losses), as a mechanism for evaluating decisions. The framework applies adaptive and predictive control engineering methods to dynamically adjust to changes in the battle, taking into account the capabilities and maneuvers of the adversary and the effects produced. Also, it includes a learning mechanism to improve decisions under conditions with high uncertainty.

The thesis reports the empirical evaluation of the framework on the Battle of Crete, Iwo Jima, and Kursk, three influential World War II battles, where the type of combat was mainly land-based. This mode of combat has not essentially changed since then. Therefore, the collected experimental results can be extrapolated to present-day land combat. This, by itself, constitutes a relevant contribution, as most literature on military decision-making lacks adequate experimental validations.

Finally, this thesis provides practitioners and researchers with guidance on the available literature, identifying the strengths and weaknesses of existing decision-making models, and giving a reference background for applying battle prediction models in decision-making.

Keywords: Decision support systems, Combat models, System dynamics, Battle situation, Warfare information system, Lanchester models, and War games.

Resumen

Los ejércitos siempre han sentido la necesidad de basar sus decisiones en métodos de investigación operativa de probada eficacia, con la que proporcionar al mando alternativas en el proceso de decisión, abarcando desde la optimización de las operaciones hasta la evaluación estratégica y el coste económico de las mismas.

Las bajas de efectivos y material en la batalla son objeto de estudio en la investigación de operaciones militares, que aplica modelos matemáticos para cuantificar la probabilidad de victoria *vs.* las pérdidas. En particular, se han propuesto diferentes formas de modelar el curso de las batallas, pero ninguno de ellas ha proporcionado un soporte adecuado para la toma de decisiones de planas mayores. Para superar esta situación, esta tesis propone un nuevo enfoque que supera la mayoría de las limitaciones de los modelos tradicionales y apoya la toma de decisiones en los niveles más altos del mando: el estratégico y el operacional, recurriendo a la determinación de decaimiento de los niveles de las fuerzas de combate, comúnmente denominado *desgaste* (pérdidas), como mecanismo de evaluación de las decisiones. El enfoque aplica métodos de ingeniería de control adaptativo y predictivo que ajusta dinámicamente los cambios en la batalla, teniendo en cuenta las capacidades y maniobras del adversario y los efectos que producen. Además, incluye mecanismos de aprendizaje para mejorar las decisiones en condiciones de alta incertidumbre.

En esta tesis se desarrolla la evaluación empírica del nuevo enfoque en las batallas de Creta, Iwo Jima y Kursk, tres influyentes batallas de la Segunda Guerra Mundial, en las que el tipo de combate era principalmente terrestre, modo de combate que no ha cambiado sustancialmente desde entonces. Por lo tanto, los resultados experimentales deberían extrapolarse adecuadamente al combate terrestre actual, esto, por sí mismo, constituye una contribución relevante, debido a que la mayoría de la literatura relacionada con los modelos de toma de decisiones militares carecen

de las validaciones experimentales adecuadas.

Por último, esta tesis pretende orientar a los profesionales e investigadores sobre la literatura disponible, identificando los puntos fuertes y débiles de los modelos de toma de decisiones existentes, muy útiles para proporcionar una base de referencia para la aplicación de los modelos de predicción de batalla.

Palabras clave: Sistemas de apoyo a la decisión, Modelos de combate, Dinámica de sistemas, Situación de batalla, Sistemas de mando y control, Modelos de Lanchester y Juegos de guerra.

Acknowledgements

All for my family, thanks to my wife María and my daughter Lucía who always believed in me.

Gerardo Minguela-Castro
Madrid
November 2021

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1 Introduction

This chapter starts motivating this thesis and setting its scope. Then, its objectives, hypotheses, and the followed methodology are presented. Finally, a thesis outline is given, providing a brief summary of each of the remaining chapters.

1.1 Motivation

Lanchester's seminal work [Lan16] on battle dynamics' modeling has inspired significant research on the development of combat abstractions to support military decision-making under uncertainty, pursuing how to achieve superiority in combat. Lanchester's original model and its distinct evolving extensions have dominated the dynamic assessment of conventional land force balance for a long time, e.g., [WFW95, Tay79, AGK11, Off80, Kir85, SOR+09, SJ16], being used by major organizations, e.g., the US Army, the Office of the Secretary of Defense, etc., to assess a wide variety of issues, e.g., evaluating the balance of operation theater, guiding decision on weaponry choices ([Cha16]), etc.

Nevertheless, it is worth noting that Lanchesterian models have important limitations, e.g., they perform an over-simplistic one-side treatment without taking into account the opponent's capabilities, and they cannot be used for disaggregated engagements.

Literature on battle decision-making modeling has grown considerably, encompassing different Lanchester ramifications, e.g., [Dei62, SG13, Sch12, SN14], and alternative non-deterministic approaches that take advantage of modern processors' computational power. There are numerous attempts and ways to model the course of the battles but little empirical verification of these efforts. In most cases, models are built on a sand base that generates confusion about models and their real capabilities and limits. The main factors contributing to the confusion in current combat modeling are inadequate models and tools, poor verification and evaluation at best, and dissonance among experts and communities. Against this background, understanding (i) the military's decision-making process, (ii) the approach to its application, and (iii) the pieces of evidence that have supported the performance of the previous models are critical to eliminating the shaky foundation in the development of a new, consistent and reliable decision-making model.

(i), which define the applicable strategy and the evaluation and selection of the different possible courses of action (COAs), with the following guideline; *Accomplish the assigned mission at a reasonable cost*.

Decision-making is divided into levels, each echelon of military command makes decisions at its level of command, based on two main factors: *Mission* (Targets) and *Situation* (available resources, Enemy, Terrain, etc.). Inside the main aspects of the *Situation*, it is the analysis of the capabilities of enemies; Knowing their size, their deployment, and their strength.

(ii)The military command makes decisions using tools that support the establishment at different organizational levels. For the correct treatment of the models that support the tools, must use variables representative of the level of command to become model useful decision aids, thus can be classified according to their resolution;

- High, where the primary military objects are represented. They are designed to lower echelons in decision-making.

- Low, where a set of military assets, i.e., units, more or less aggregated, are represented. They are designed for higher levels of decision-making.

(iii) Typically, decision-making models are validated with mathematical procedures that make non-realistic assumptions, e.g., [Che03], simplistic made-up examples, e.g., [Kre20, Cou19], data coming from logs gathered from training sessions with combat simulators, e.g., [IHJ19] and validations against historical military data, e.g., [SD15, RC16, MPW16]. In most cases, models have not been validated adequately.

Interestingly, most decision-making approaches, including the non-Lanchesterian ones, are focused on the tactical level of command. In other words, the operational and strategic levels of command are insufficiently supported by existing decision-making systems.

1.2 Scope of this thesis

This research will analyze the current state of decision models applied to the battlefield, focusing on understanding what types of decisions are applied, how those decisions are made, and what empirical evidence supports them, which will provide insight into the limitations of current approaches and will allow proposing new mechanisms to overcome them. In this sense, our research will propose an approach to close the gaps in the automation of high-level decisions on the land battlefield, known as the strategic and operational military decisions. The validity of the proposed approach will be demonstrated by a sufficiently broad set of empirical evidence, all of which must be representative.

To understand the theoretical background in which the research is properly situated, a broad explanation of the context is provided.

1.3 Aims of the thesis

Keeping in mind the situation described presented in the introduction, the following goals have been defined:

- Eliminate the limitations of Lanchester's classic work and other Lanchester ramifications on the land battlefield.
- To provide a framework for automation of strategic and operational military decisions.
- To provide empirical evidence showing that framework fits battle trends adequately and can select the most appropriate decision.

- To guide practitioners and researchers on the strengths and weaknesses of existing decision-making models.

1.4 Research questions

This research is designed to analyze the performance of the application of control theory in the Lanchester battle decision models, in pursuit of an approach to strategic and operational decision-making in the field of land armies. On this basis, the following Research Questions (RQs) have been considered:

- **RQ1**: Are the operational and strategic levels of command sufficiently supported by existing decision-making systems?
- **RQ2**: Can adaptive and predictive control architecture contribute to overcoming the limitations of traditional battle models?

1.5 Hypotheses

Concerning **RQ1**, it is expected that the operational and strategic command levels are not sufficiently supported by existing decision-making systems, even the new models that are gaining momentum, do not provide an adequate reference point for high-level decision-making.
Regarding **RQ2**, our approach, which applies adaptive and predictive control theory and incorporates uncertainty modeling techniques, is expected to overcome the limitations of traditional models by treating the battle as

a cause-effect process that evolves according to the dynamics of the Lanchester's equations subject to changes and external actions.

1.6 Methodology

The research was carried out following the usual scientific methodology:

- Wording of hypotheses.
- Collection of observations.
- Contrast between the hypotheses and the observations.
- Readjustment of the initial hypotheses in the light of the obtained results.

1.7 Personal motivation

This thesis has been motivated by the fact that my career has been devoted to the design of constructive simulators in which a gap was detected in the automation of high-level decisions of the opposing force, known as strategic and operational military decisions, to which this work aims to answer.

1.8 Thesis outline

This thesis is organized into **five chapters**.

Chapter II summarizes related work, identifying the strengths and weaknesses of existing decision-making models, and contains a comprehensive Literature review.

Chapter III presents this thesis's main contribution: an innovative framework that overcomes most limitations of Lanchesterian models and supports decision-making at the highest command levels, where the design decisions and construction details are broadly justified and explained in-depth.

Chapter IV contains the bulk of the empirical analysis, reporting the results of the implementation in the Battle of Crete, Iwo Jima, and Kursk, which are among the largest battles of World War II.

Chapter V contains the primary conclusions and future works.

In addition, **four appendices** cover some issues and calculation procedures to complete this thesis' understanding.

Appendix A summarizes the most prominent Lanchester's equations for combat.

Appendix B traces the relationship between the defender's advantage factor parameter and the probability of rejecting the enemy's attack using the logistic regression method.

Appendix C describes the calculation procedures used in solving Ordinary Differential Equations (ODE).

Appendix D shows the initial feasible values and set-points, which are the basis for control actions.

1.9 Academic contributions

After the time invested researching and studying the related literature, the contribution to this field of knowledge, besides this thesis, has been a novelty framework to automation of strategic and operational military decisions focused on dynamically adjusting the factors that define the evolution of the battlefield, including learning mechanisms that optimize the capabilities of the architecture and the ability to improve decisions under uncertainty.

Finally, a guide for practitioners and researchers on the available literature has been emerged from the literature review, very useful to provide a reference background for the application of battle prediction models in decision-making.

In parallel, this thesis has generated the following related **journal articles**:

1. Gerardo Minguela-Castro, Ruben Heradio, and Carlos Cerrada. Automated Support for Battle OperationalâĂŞStrategic Decision-Making. *Mathematics*, 2021, 9(13), 1534. https://doi.org/10.3390/math9131534

2. Gerardo Minguela-Castro, Ruben Heradio, and Carlos Cerrada. Automated Support for Battle Decision-Making: a Systematic Literature Review. *Military Operations Research (MOR)*, 2021. Accepted and waiting for being published.

and the following **conference papers**:

1. Gerardo Minguela-Castro, Carlos Cerrada and José A. Cerrada. Estudio del modelo de combate de Lanchester como soporte para la construcción de un decisor estratégico operacional militar mediante bloques retroalimentados. *XL Jornadas de Automática*, pp.528-534, La Coruña, Spain, 2019. https://ruc.udc.es/dspace/handle/2183/23789

2. Gerardo Minguela-Castro, Carlos Cerrada and José A. Cerrada. Decisor Estratégico Operacional Militar mediante bloques retroalimentados, utilizando técnicas de modelización de la incertidumbre. *XV Simposio CEA de Control Inteligente, V Simposio CEA de Modelado Simulación y Optimización*, La Rioja, Spain, 2019. https://www.researchgate.net/publication/340528931_Military_Operational_Decision_Maker_based_on_Blocks_feedback_using_uncertainty_techniques

1.10 Materials

Following open science's good practices, the different software artifacts associated with the empirical validations are available publicly at this GitHub repository:

https://github.com/gminguela/the-empirical-validation-of-the-framework-on-the-Battle-of-Crete-Iwo-Jima-and-Kursk-and-others

All the materials were developed with Excel spreadsheets through Microsoft Excel 2016 on tables that define the battle events based on attrition values. The following Excel plug-ins have also been used; EZAnalyze v3.0, XLSTAT 2021 v1.4, and Simple Decision Tree v1.4.

2

Related Work

This chapter summarizes related work in order to answer RQ1. First, Section 2.1 decomposes RQ1 into three more detailed questions. Then 2.2 presents the procedure followed to perform the literature review that enables answering RQ1 systematically. Section 2.3 presents the results of the review and answers RQ1. Finally, Section 2.4 summarizes this chapter conclusions.

2.1 Research Questions

RQ1 can be broken down into the following fine-grained questions:

1. *What sort of decisions do existing models support? (RQ1.1)*. In military doctrine, three levels of command are typically distinguished according to their aggregation degree, [NAT11]: strategic, operational, and tactical. Our review not only identifies the type of decisions that available models support, but also to which aggregation level they correspond.

2. *How do those models work?, i.e., What is their theoretical basis? (RQ1.2)*. Models and decision-making techniques can be classified into deterministic and non-deterministic. This review shows that approx-

imately 70% of the literature follows the Lanchesterian deterministic theory, e.g., [Eng54, Hel65, Pet67, Bra95]; nevertheless, non-deterministic approaches based primarily on stochastic mechanisms, e.g., [HM00, KT99] and intelligent agents are gaining substantial momentum, e.g., [ADK17, OT17, LS02, Ila00].

3. *To what extent have models been validated? (RQ1.3)*. This review shows that, in most cases, models have not been validated adequately, as there is a lack of empirical evidence supporting models' performance. Typically, models are validated with (i) mathematical procedures that make non-realistic assumptions, e.g., [McN99, Che03], (ii) simplistic made-up examples, e.g., [SG06, TB78, XZ14, Che03], or (iii) data coming from logs gathered from training sessions with combat simulators, e.g., [IFW91, LS02, TYM00]. Only a few articles report model validations against historical military data, e.g., [Eng54, Pet67, Bra95]. In this regard, this review collects historical datasets to support future model evaluations.

2.2 Associated literature review methodology

The review was conducted following the guidelines given in [KC07, WRH+12], which encompasses three main activities: (i) identification of the literature of interest, (ii) its analysis and classification, and (iii) the synthesis of the results.

2.2.1 Literature search strategy

Figure 2.1 depicts the procedure adopted for collecting the literature this paper reviews. Gathering the total population of articles that fall into the scope of a literature review is usually unfeasible [WRdMSN+13]. Hence, We endeavored the more pragmatic goal of getting a publication sample that described the population adequately. To do so, the query in Figure 2.2 was run on April 24th, 2020, over three databases: Clarivate Analytics-Web of Science (WoS), Elsevier Scopus, and the Defense Technical Information Center (DTIC).

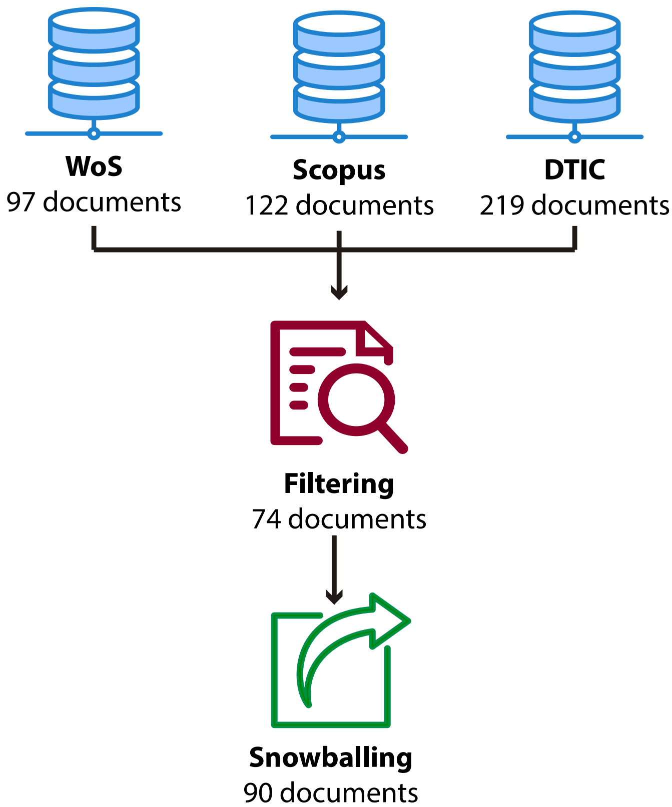

Figure 2.1: Literature identification process.

WoS and Scopus were selected because, according to various studies [VG09, GJCJMO14, SL18], they render the highest quality bibliographic data for longitudinal literature reviews. DTIC was chosen because it is the official repository of the United States Department of Defense, and thus it includes some relevant documents for our review, particularly technical reports.

The query was refined successively until a favorable balance between completeness and avoidance of false-positives was accomplished. The query in Figure 2.2 follows the WoS notation. `TS` expresses that the search is performed on the papers' *topic*; that is, considering their title, abstract, and keywords. The wildcard `*` looks for plurals and word-inflected forms. `AND`, `OR`, and `NEAR` set relationships between terms. In particular, w_1 `NEAR/0` w_2 means that between the words w_1 and w_2 there cannot be an additional word; e.g., `war NEAR/0 game*` catches "war game", "war-game", "war games", etc.

```
TS = (
 ( combat OR  battle  OR  militar*  OR  war )  AND
 ( model* OR simulation* OR war NEAR/0 game*  OR  decision NEAR/0 making  OR
   decision NEAR/0 support OR operational NEAR/0 research )  AND
 ( attrition OR weakening OR "grinding down" OR reduction OR
   abatement OR decrease OR attenuation ) AND
 ( lanchester* OR attrition NEAR/0 stochastic NEAR/0 model* OR
   markov NEAR/0 chain OR effect* NEAR/0 based NEAR/0 operation* OR
   "course of action" OR "battle plan" )
)
```

Figure 2.2: Query used to retrieve the publication sample from WoS, Scopus, and DTIC.

2.2.2 Literature filtering

The documents extracted from WoS, Scopus, and DTIC were refined according to precisely defined inclusion and exclusion criteria. In particular, Inclusion (I) was supported by the following criteria:

I1 Documents about the definition, optimization, historical validation, or performance analysis of battle models.

I2 Documents on the automated support for battle decision-making.

I3 Documents that contribute to ask at least one of these review RQ1.s.

The next criteria were used for Excluding (E) studies:

E1 Documents not written in English (except for authors of this text).

E2 Documents not accessible in full-text.

E3 Documents out of the military scope.

For filtering the publications, each paper was judged according to the inclusion/exclusion criteria above. A document passed the filter if it satisfied at least one of the inclusion criteria and none of the exclusion criteria.

After filtering the sample, there were 74. Then, the *snowballing* guidelines given in [Woh14] were followed to find extra relevant articles by examining the references of the papers gathered from the databases. As a result, a sample of 90 documents was obtained.

2.3 Results and discussion

2.3.1 Publication sample overview and principal scientific actors

Figure 2.3 shows the distribution of the sample according to (i) whether the documents' origin is civil or military and (ii) their source type, distinguishing between journal articles, technical reports, Ph.D. theses, and books. 46.67% and 53.33% of the papers originate from civilian and military institutions, respectively. 52.22% of the documents have been published in journals. Figure 2.4 summarizes the top five journals that have published the largest number of articles.

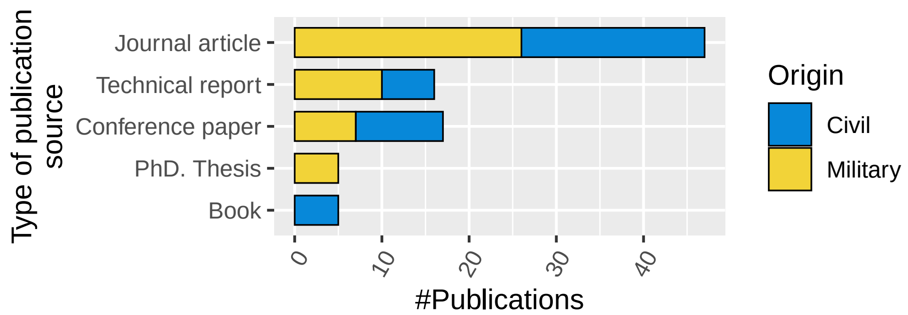

Figure 2.3: Document sample distribution according to their civil/military origin and publication source type.

Figure 2.5 shows the institutions that have published the most literature (43.33% of all documents). Figure 2.6 depicts the document distribution per country. It is worth noting that, due to the confidentiality policies that limit what can be published in each country, the sample lacks documents from some relevant actors in military terms, such as Russia.

Figure 2.4: Most prolific journals.

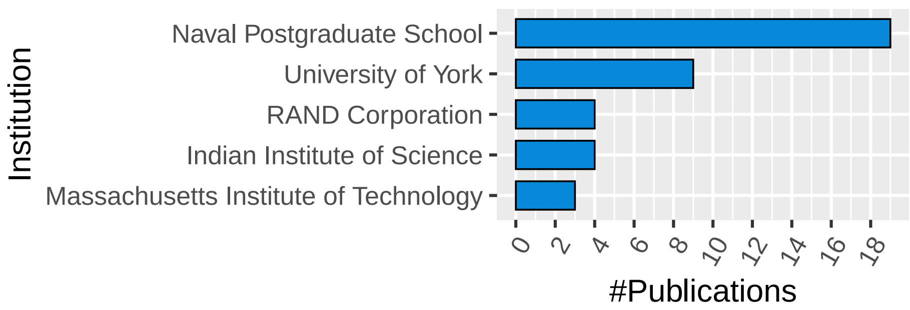

Figure 2.5: Most prolific institutions.

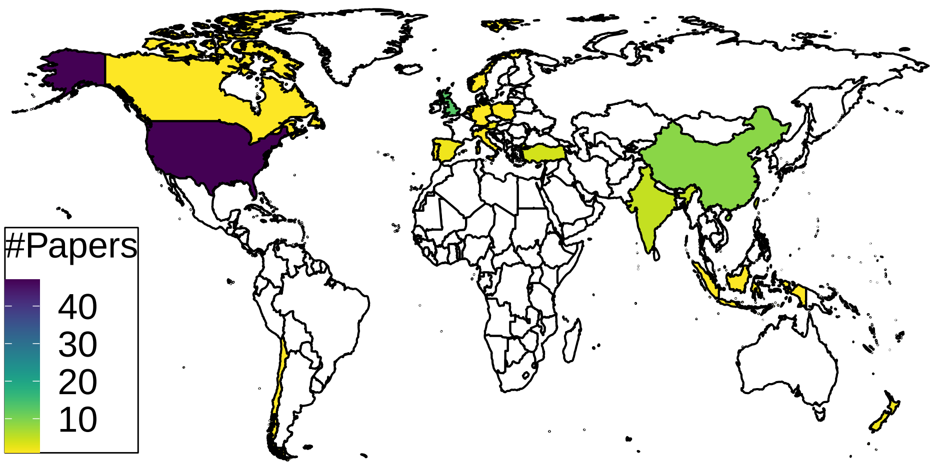

Figure 2.6: Publication distribution across countries.

2.3.2 RQ1.1: What sort of decisions do existing models support?

There are three main uses for battle decision-making models:

1. *Description.* The model's purpose is to describe combat situations or battles that have already happened, improving the understanding of the decision consequences.

2. *Prediction.* The model's aim is to predict the battle evolution and thus anticipate potential difficulties.

3. *Training.* The model's purpose is to educate military personnel.

Furthermore, models support the decision at the four command levels typically considered in military theory:

1. *Strategic level.* It studies the conflict from the most abstract perspective, considering the war's final outcomes as a whole. It involves the overall planning, resource distribution, and organization of the military force. Also, it defines and supports the national policy.

2. *Operational level. War* is divided into *campaigns*, which are organized into *operations*. The operational level deals with the design, arrangement, and execution of campaigns and principal operations.

3. *Tactical level.* It implements the campaign operations on the battlefield.

4. *Execution level.* It deals with the fulfillment of the duties defined at the tactical level.

Table 2.1 distributes the document sample according to the command level where papers are focused. References are sorted in descending order by their number of citations. The notation [reference]$_{\#\text{citations}}$ is used, e.g., [Eps88]$_{23}$ means that [Eps88] has been cited 23 times since its publication. Some documents appeared in various databases with a different number of citations; in those cases, Table 2.1 displays the maximum value. It is worth noting that, above all articles, Lanchester's seminal work [Lan16] stands out with 87 citations.

Command level	**References**
Strategic	[Eps88]$_{23}$, [Mac15]$_{11}$, [Sch07]$_{4}$, [DDF^{+}16]$_{3}$, [MTL07]$_{3}$
Operational	[Tol12]$_{174}$, [WK09]$_{115}$, [Bra95]$_{102}$, [Eps85]$_{98}$, [Hel65]$_{81}$, [Fri98]$_{74}$, [LT04]$_{66}$, [HH95]$_{57}$, [Luc00]$_{56}$, [PSD89]$_{44}$, [JM15]$_{38}$, [KT99]$_{33}$, [Rod89]$_{27}$ [R72]$_{26}$, [WPY00]$_{25}$, [LD04]$_{22}$, [Din01]$_{22}$, [Hel61]$_{19}$, [Eng54]$_{18}$, [Che03]$_{15}$, [CJLL12]$_{13}$, [Was00]$_{11}$, [HM00]$_{10}$, [Duf17]$_{8}$, [Che07]$_{8}$, [RC16]$_{7}$, [Sch12]$_{6}$, [Tam98]$_{5}$ [YHJ13]$_{4}$, [Hel97]$_{3}$, [MTL07]$_{3}$, [Tol16]$_{2}$, [You72]$_{2}$, [Cou19]$_{1}$, [CQ14]$_{1}$, [HML17]$_{0}$ [Zha15]$_{0}$, [Kle80]$_{0}$
Tactical	[Lan16]$_{876}$, [Dei62]$_{174}$, [Tol12]$_{174}$, [WK09]$_{115}$, [Eps85]$_{98}$, [Sch67]$_{69}$, [IFW91]$_{62}$, [Luc00]$_{56}$, [Dav95]$_{54}$, [PSD89]$_{44}$, [JM15]$_{38}$, [JM11]$_{36}$, [KT99]$_{33}$ [HD87]$_{32}$, [Pet67]$_{30}$, [Rod89]$_{27}$, [TYM00]$_{26}$, [GV11]$_{22}$, [KCF^{+}18]$_{20}$, [McN99]$_{14}$, [Che03]$_{15}$, [SG06]$_{14}$, [CJLL12]$_{13}$, [KLM18]$_{12}$, [TBBM01]$_{11}$, [Was00]$_{11}$, [KM14]$_{11}$ [SG05]$_{11}$, [TBBM01]$_{11}$, [Kir85]$_{9}$, [TV03]$_{9}$, [KBG^{+}05]$_{8}$, [Che07]$_{8}$, [Joh96]$_{7}$, [RC16]$_{7}$, [Tay82]$_{6}$, [KLM18]$_{5}$, [WSWL12]$_{3}$, [You72]$_{2}$, [Tol16]$_{2}$, [AS15]$_{2}$, [JZD17]$_{2}$, [Cou19]$_{1}$, [CQ14]$_{1}$[JHC17b]$_{1}$, [Wan14]$_{1}$, [Kle80]$_{0}$, [Kre20]$_{0}$, [HML17]$_{0}$
Execution	[Ila00]$_{255}$, [Tol12]$_{174}$, [LS02]$_{138}$, [Luc00]$_{56}$, [WK09]$_{115}$, [PSD89]$_{44}$, [SG08a]$_{19}$, [Mac09]$_{19}$, [LM14a]$_{14}$, [McN99]$_{14}$, [GJ97]$_{11}$, [ML08]$_{7}$, [And93]$_{6}$ [JYL^{+}18]$_{5}$, [SG08b]$_{3}$, [You72]$_{2}$, [Tol16]$_{2}$, [BS13]$_{2}$, [ADK17]$_{0}$, [IHJ19]$_{0}$

Table 2.1: Document sample distribution according to their command level.

Figure 2.7 depicts the distribution of the document sample according to their command level and purpose. The majority of the models have a descriptive and predictive use in the operational and tactical levels, i.e., the literature mostly focuses on the intermediate command levels. Only 4.44% and 18.9% of the papers are concerned with the most abstract and concrete levels, respectively. On the one hand, strategic decisions involve issues challenging to model, e.g., intuition; on the other hand, the execution level is usually fraught with details that make models extremely complicated.

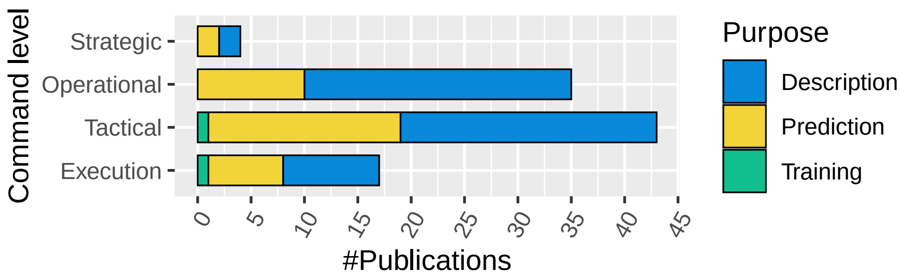

Figure 2.7: The number of publications according to their command level and purpose.

2.3.3 RQ1.2: How do those models work?

Table 2.2 summarizes the trends found in the literature review on battle modeling, focusing on Lanchesterian models but with others emerging lately, such as Agent-Based Models (ABM).

Introduction to Lanchester models

Lanchester hypothesized that in modern war the concentration of forces was an appropriate tactic. To demonstrate this, developed the Lanchester mathematical equations that simplify the battle models, pointing calculating attrition of forces up in the military engagement result. These equations can be found in Appendix A and their evolution, as well as their further discussion and study in Section 3.2.1.

Model type	References
Lanchesterian	[Lan16]$_{876}$, [Dei62]$_{174}$, [Tol12]$_{174}$, [WK09]$_{115}$, [Bra95]$_{102}$, [Lep87]$_{91}$, [Hel65]$_{81}$, [Fri98]$_{74}$, [Sch67]$_{69}$, [LT04]$_{66}$ [HH95]$_{57}$, [Mac06]$_{52}$, [Dav95]$_{54}$, [PSD89]$_{44}$, [JM15]$_{38}$, [JM11]$_{36}$, [HD87]$_{32}$, [Pet67]$_{30}$, [R72]$_{26}$, [TYM00]$_{26}$, [LD04]$_{22}$ [Din01]$_{22}$, [SBB15]$_{22}$, [GV11]$_{22}$, [KCF+18]$_{20}$, [Hel61]$_{19}$, [SG08a]$_{19}$, [Mac09]$_{19}$, [Eng54]$_{18}$, [Che03]$_{15}$, [SG06]$_{14}$, [LM14a]$_{14}$ [CJLL12]$_{13}$, [GJ97]$_{11}$, [SG05]$_{11}$, [Was00]$_{11}$, [KM14]$_{11}$, [TB78]$_{10}$, [Kir85]$_{9}$, [Che07]$_{8}$, [Duf17]$_{8}$, [Joh96]$_{7}$, [Sch12]$_{6}$ [Tay82]$_{6}$, [KLM18]$_{6}$, [Tam98]$_{5}$, [JYL+18]$_{5}$, [YHJ13]$_{4}$, [MTL07]$_{3}$, [SG08b]$_{3}$, [Hel97]$_{3}$, [JZD17]$_{2}$, [AS15]$_{2}$, [You72]$_{2}$, [Tol16]$_{2}$ [JHC17b]$_{1}$, [CQ14]$_{1}$, [Cou19]$_{1}$, [Kle80]$_{0}$, [DDF+16]$_{0}$, [Sym17]$_{0}$, [Kre20]$_{0}$ [HML17]$_{0}$, [Zha15]$_{0}$
Markovian	[WK09]$_{115}$, [Mor48]$_{21}$ [KCF+18]$_{20}$, [McN99]$_{14}$, [HM00]$_{10}$, [BS13]$_{2}$, [JHC17a]$_{2}$
Stochastic	[Tol12]$_{174}$,[IFW91]$_{62}$,[Sch07]$_{4}$,[And93]$_{6}$,[Tol16]$_{2}$
Cellular Automata/Agent-Based Models	[Ila01]$_{959}$ [Ila00]$_{255}$, [LS02]$_{138}$, [TV03]$_{9}$, [WSWL12]$_{3}$, [Wan14]$_{1}$, [ADK17]$_{0}$
Bayesian Models	[WPY00]$_{24}$, [RC16]$_{7}$
Epstein Models	[Eps85]$_{98}$ [Eps88]$_{23}$
Learning Agent Shell	[TBBM01]$_{11}$
Fractals	[ML08]$_{7}$
Entropic	[Rod89]$_{27}$
Random Forests	[IHJ19]$_{0}$

Table 2.2: Document sample distribution according to how battle dynamics are modeled.

Shortcomings of Lanchester models

Despite the experimental validations that the literature depicts, the criticisms that Lanchester models often face, such as those of [Lep87] and [Eps85], are shown in large groups, including some nuances and considerations from the reviewed literature that deepen and clarify.

- Fitting the battle data is not good, [WPY00].

  It should be noted for the correct treatment of the Lanchester models that in most cases of documented battles, the daily casualty data are not known accurately, and the assessments made must be taken with care. Thus, in the fitting of data of battles, it is usually affected by;

  - Treatment of large battles with multiple types and phases as a whole. The sequential factors of the evolution of the battle, change the *lethality*, or more generalist Lanchester models that reach more types are not taken into accounts, [HH95], [LD04], [Bra95] and [McN99].

– Using a constant *lethality* factor. The factor is dependent on time and events, see [TB78]. This way, the evolution of the battle, fatigue, or motivation must be taken into account, [LD04], [Che03], and [RC16].

- A simplistic model only treats attrition without taking into account other important factors, [KCF+18].

  It should be noted for the correct treatment of the Lanchester models that the *lethality* depending on the organization, posture, motivation or fatigue and not only values dependent on the characteristics of the weaponry, [HD87] and [MC19a]. Thus, [Sch67] used Lanchester models to the Vietnam War, concluding that the *lethality* of insurgents in the case of ambush increases, minimizing their casualties, demonstrating the dependence of other factors.

- Unilateral treatment of the model without taking into account the opponent's capacity for spatial-temporal modification, [Eps85].

  The use of adaptive control theory eliminates unilateralism, adapting the model to the previous opponent spatial-temporal decisions made that modify the *lethality*, [MC19a].

- The law of diminishing marginal returns, Lanchester's Square Law denies this law that affects all social processes, [Eps85].

The use of adaptive control theory allows the modification of the *lethality* in the course of the battle. Therefore, the increase in the number of forces to the point of being a crowd will affect their performance in battle, this consequence will be collected by the adaptive mechanism, [MC19a].

- Disaggregated models. Lanchester models appear to perform unsatisfactorily for small groups, [Mor48].

  It should be noted for the correct treatment of Lanchester models that the models must use variables representative of the level of command to become model useful decision aids. The level of application of Lanchester models is in line with levels of strategic-operational aggregation, [MC19a, MC19b], although current combats seem very sophisticated Lanchester models determine it, this is particularly true in highly aggregated models, [You72] and [McN99].

  More aggregated models cover up the most basic execution mechanisms of the battle, such as individual clashes. On the other hand, lower levels of aggregation are likely to be affected by factors such as weaponry, position, visibility, logistics, etc., [Was00] and [PSD89]. At the execution level, details will come in that will make the mathematical modeling very complex. Stochastic models are usually used in these cases, [Kle80].

  Figure 2.8 depicts the type of decisions supported by the models, according to the level of aggregation and abstraction, thus a Hierarchy of Models emerges from the literature review.

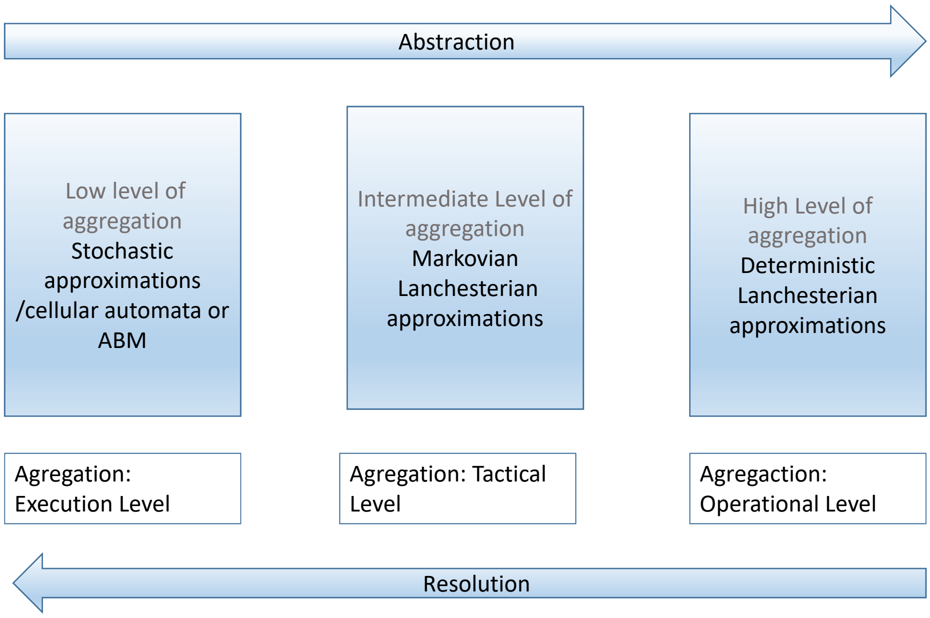

Figure 2.8: Type of decisions supported by the models. Aggregation level vs. Abstraction.

Alternatives to the Lanchester models

In the literature analyzed, two common mechanisms of battle analysis are found, through stochastic models and deterministic models, some of them from the Lachesterian tradition. These Lanchester alternatives look for expanding the capabilities of existing models and reduce shortages, and have been based on:

- Incorporating random effects of attrition.
- Considering factors affecting *lethality* ([HPB+91]), e.g., time, position, force scale, training, skills, etc.
- Incorporating other non-battle attrition.
- Extending to heterogeneous forces.

- Adding Intelligence and technology factors.
- Scaling up or down4 the level of resolution.
- And others of similar consideration.

In the usual sources, multiple alternative approaches are cited, [Mor48], [Eps85], [Lep87], [Rod89], [PSD89], [ML08], etc. There are numerous attempts and ways to model the course of the battles, but little empirical verification of these efforts, in many cases models are built on a sand base, [DB91]. Table 2.3 shows the main alternative approaches, finding out from the literature review vs. the shortcomings of theirs implementation.

It is worth highlighting some studies available in the associated literature, which are representative of the innovation of the approach itself, or the modeling method.

- [Eps85] incorporated in the attrition models, the spatial-temporal relationship of the fighters as a modifier of the *lethality* in operations. So, the feedback of the defensive and offensive maneuver affects the attrition between fighters.
- [PSD89] developed the space-time relationship that does not cover the ordinary Lanchester models. In their approximation, the equations in partial derivatives of the model add as an independent term not only time but also space.
- [Rod89] developed the relationship between the entropy of Shannon's information and the degradation of forces in battle, thus es-

tablishing a breakpoint of 37% that matches with the maximum of entropy.

- [KT99] modeled the evolution of the battle as a Markov chain, a particular case of the stochastic process. In addition, [HM00] developed the Kolgomorov equations on a Markovian process and compared Markovian and Lanchesterian models, finding no difference in the large-scale command level.

- [ML08] introduced the fractal concept into the attrition assessment, which is effective in encapsulating complex aspects of the battle. The chaotic behavior in certain phases of the battle is not supported by the deterministic models of Lanchester.

- [JM15] incorporated Richardson's models of the arms race into Lanchester's models, as a focus in the study of the insurgency.

- Computational models among which are cellular automata, [Ila00, Ila01], and their generalization agent-based models, [LS02, ADK17], have appeared for the exploration of combat as a complex self-organizing adaptive system, bottom-up approach. They are essentially complementary to the Lanchester models, [Tol12], and should be used for the understanding of behaviors that appear out of balance, chaotic phases, [Ila01].

		Alternative approaches to the Lanchester models from the literature			
		Epstein model	**Stochastic models**	**Markovian Lanchesterian approximations**	**Cellular Automata/ ABM**
Shortcomings	**A simplistic model**	X			
	The opponent's capacity for spatial-temporal modification		X	X	
	Duels	X		X	
	Disaggregated models	X			X
	Fitting the battle data	X	X		X
	The law of diminishing marginal returns		X	X	

Table 2.3: Alternative approaches vs. shortcomings

Other techniques from Artificial Intelligence are useful to model the chaotic behavior in certain phases of the battle, and at the aggregate level to determine patterns and heuristics based on the statistics of the combat model, which should allow obtaining rules and knowledge for the most effective decision-making.

How are the decision-making models applied?

Decision-making models have been widely used for battle analysis and evaluation, from fire optimization to strategic assessment or battle economic cost. The classic applications from *Operational Research*, e.g., [MK46, BKR95, Arm14], are listed, giving an overview of its impact on military decision-making:

- The evaluation of the Course of Actions (COAs).
- Theater of operations scenario analysis.
- Relationship between intelligence and attrition for the increment of precision in decision-making models, as well as the optimization of war efforts.
- Fire Optimization on heterogeneous fighters for the maximization of enemy casualties and minimization of reinforcements.
- Computer Assisted Exercises (CAX), simulation of the operational background for training in leading and operation.
- Economic cost calculation and force replacement.
- And others of similar consideration.

Representative examples of application that allows us to realize how decision-making models are applied, not included so far, can be identified in the literature review:

- [SG08a] used the Square Law for the optimization of the defensive tactics associated with the disposition and distribution of the resources.
- [DDF+16] evolved from Kress and Szechtmann's model for government decision-making in the fight against insurgency, trying to optimize the relationship between intelligence in the interaction with insurgents, taken as a control variable in the application to intelligence effort, recruitment, against measures that minimize casualties.
- [Joh96] quantified the Lanchester models based on the knowledge of the enemy's COA.
- [Cou19] demonstrated through the Lanchester models that military intelligence is a factor multiplier that can compensate for force superiority.
- [MTL07] analyzed the strategic factors 'undermining factors and capital decapitation' in the course of battle.
- [CJLL12] studied the strategies of minority attacks.
- [Mac09] optimized direct fire distribution on heterogeneous combat units and [LM14b] set an optimal fire distribution between homogeneous and heterogeneous forces for maximizing enemy casualties and minimizing reinforcements, all through the Lanchester models.

2.3.4 RQ1.3: To what extent have models been validated?

The contrast experimental defines the ability of battle models to predict the attrition, battle data are an intrinsic difficulty in validating models which, as far as possible, suffers from errors of perception resulting from two fundamental factors: The opponent's data is not always available and the plurality of sources.

Data sources	Internet address
R. L. Helmbold", Historical Data and Lanchester's Theory of Combat,"CORG-SP-128	https://apps.dtic.mil/dtic/tr/fulltext/u2/480975.pdf
Willard D. (1962) 'Lanchester as a force in history: An analysis of land battles of the years 1618-1905'	https://apps.dtic.mil/dtic/tr/fulltext/u2/297375.pdf
Livermore T.L. (1900) 'Numbers and Losses in the Civil War in America, 1861âĂŞ65'	https://academic.oup.com/ahr/article-abstract/6/3/598/58455? redirectedFrom=fulltext
CDB90 dataset of individual battles, 1600-1979	https://github.com/jrnold/CDB90
Evolution of Modern Battle: an Analysis of Historical Data. School of Advanced Military Studies	https://apps.dtic.mil/sti/pdfs/ADA233235.pdf
Kursk Operations Simulation and Validation Exercise Phase II	https://apps.dtic.mil/dtic/tr/fulltext/u2/a360311.pdf
The Ardennes Campaign Simulation Database	https://apps.dtic.mil/sti/citations/AD1034106
Capt Clifford P. Morehouse, The Iwo Jima Operation, USRICR, Historical Division, Headquarters U. S. Marine Corps, 194	https://www.worldcat.org/title/iwo-jima-operations/oclc/9450395
Testimony of GEN. MAXWELL D. TAYLOR before House Appropriations Subcommittee on Defense Appropriations, 1960	not found

Table 2.4: Data sources of historical battles

How sources use different methods of validation is a fundamental question, finding out the quality of the hypotheses reached. Thus, in the literature review has been found the use of heterogeneous sources as a validation and testing mechanism, grouped below:

- Historical texts, testimonies, and Data-Sets.

Data sets do not have a distinction by days, phases, types, etc. in any case. This involves that authors perform middle data, based on reading historical sources or assumptions. Table 2.4 contains the data sources for the literature review.

- (iii) Data coming from logs gathered from training sessions with combat simulators (WarGames).

Performing this task requires a deep knowledge of the simulation system, knowledge is not always complete. In addition, combat simulators are designed for one level of aggregation and resolution, so scaling data up and down, as is sometimes the case, generates wrong data for analysis, [Dav95]. Similarly, simulation exercises are intended for training purposes and not necessarily to obtain data for analysis. For all these reasons, simulation data from training sessions are a rather dubious way of testing, [IFW91].

- The rest of the authors validate their assumptions through (i) mathematical procedures that make non-realistic assumptions or (ii) simplistic made-up examples or even scores by experts.

Figure 2.9 depicts publication distribution according to how they are validated.

Table 2.5 shows the citation classics for the different validation ways, showing, in most cases, models have not been validated adequately, as there is a lack of empirical evidence supporting models' performance.

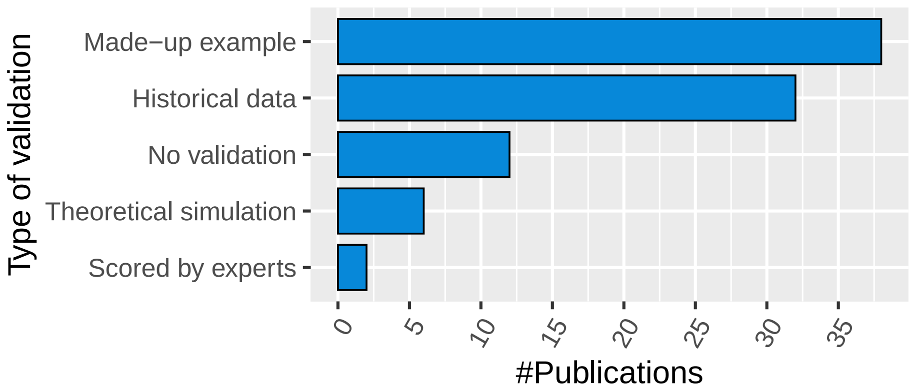

Figure 2.9: Validation and testing mechanism.

Type of validation	References
Made-up Examples	[Ila00]$_{255}$, [LS02]$_{138}$, [Eps85]$_{98}$, [Sch67]$_{69}$, [Mac06]$_{52}$, [PSD89]$_{44}$, [HD87]$_{32}$, [GV11]$_{22}$, [KCF^{+}18]$_{20}$, [SG08a]$_{19}$, [Mac09]$_{19}$ [Che03]$_{15}$, [SG06]$_{14}$, [LM14a]$_{14}$, [McN99]$_{14}$, [CJLL12]$_{13}$, [GJ97]$_{11}$, [KM14]$_{11}$, [SG05]$_{11}$, [TB78]$_{10}$, [Che07]$_{8}$, [Tay82]$_{6}$, [KLM18]$_{6}$, [Sch07]$_{4}$ [MTL07]$_{3}$, [SG08b]$_{3}$, [JYL^{+}18]$_{3}$, [WSWL12]$_{3}$, [You72]$_{2}$, [BS13]$_{2}$, [Cou19]$_{1}$, [JHC17a]$_{1}$, [Kle80]$_{0}$, [DDF^{+}16]$_{0}$, [Cou19]$_{1}$, [CQ14]$_{1}$, [Kre20]$_{0}$, [ADK17]$_{0}$
Historical data	[Lan16]$_{876}$, [Dei62]$_{174}$, [WK09]$_{115}$, [Bra95]$_{102}$, [Fri98]$_{74}$, [LT04]$_{66}$, [HH95]$_{57}$, [JM15]$_{38}$, [JM11]$_{36}$, [Pet67]$_{31}$, [Rod89]$_{27}$, [R72]$_{26}$ [WPY00]$_{24}$, [LD04]$_{22}$, [Din01]$_{22}$, [Hel61]$_{19}$, [Eng54]$_{18}$, [Was00]$_{11}$, [HM00]$_{10}$, [Kir85]$_{9}$, [TV03]$_{9}$, [Duf17]$_{8}$, [RC16]$_{7}$, [Sch12]$_{6}$, [Tam98]$_{5}$ [YHJ13]$_{4}$, [Hel97]$_{3}$, [JZD17]$_{2}$, [AS15]$_{2}$, [Wan14]$_{1}$, [Sym17]$_{0}$, [HML17]$_{0}$
Score by Experts	[TBBM01]$_{11}$ [KBG^{+}05]$_{8}$
Theoretical Simulations	[IFW91]$_{62}$ [TYM00]$_{26}$, [SBB15]$_{22}$, [ML08]$_{7}$, [Joh96]$_{7}$, [IHJ19]$_{0}$
No validation	[Ila00]$_{255}$, [Tol12]$_{174}$, [Lep87]$_{91}$, [Hel65]$_{81}$, [Luc00]$_{56}$, [Dav95]$_{54}$, [Eps88]$_{23}$, [Mor48]$_{21}$, [Mac15]$_{11}$, [And93]$_{6}$, [Tol16]$_{2}$, [Zha15]$_{0}$

Table 2.5: Sample distribution according to the validation mechanism.

2.4 Concluding remarks

In the current scenarios where the algorithms designed together with the technology allow analyzing and dissect the data from battles, the interest in using *the Decision Theory* in a military background, from different prospects, has been renewed.

The state of knowledge and the variety of research show a degree of convergence and conclusion, which is very useful to focus on debate and answers the questions of this literature review:

Thus, the first research question focuses on the command level of military decisions (RQ1.1: What sort of decisions do existing models support?), where the current precedents of automation are fundamentally focused on the leading of the battle, leaving apart decisions of a higher level of abstraction and depth, the so-called strategic-operational decisions, in this case, intuition and analysis must come together to forge an acceptable solution.

The analysis of the different decision-making solutions provides an inside view of the process (RQ1.2: How do those models work?), where Lanchester models and some of their variants provide an effective framework for predicting the evolution of warfare by predicting combat, taking into account the correct command level, the types and phases of the large battles, a non-constant *lethality* and incorporation of feedback loops and adaptive mechanisms enable models to be adjusted dynamically to developments on the battlefield. However, there is a lack of robustness in the analytical algorithms when the cause-effect relationship of the battle cannot be modeled (chaotic behavior in certain phases of the battle), techniques such as Artificial Intelligence should be useful in this case. Thus, both techniques should be considered complementary.

In the long term, the inclusion of learning mechanisms will optimize the capabilities of the models and, in the short term, the capacity to improve the decisions under uncertainty.

Finally, evidence that has supported the performance of the models is essential to establish a reliable framework of solutions (RQ1.3: To what extent have models been validated?), there have been numerous attempts to measure force and build models of combat, but beyond these efforts, there is usually no empirical verification of them with real situations even historical data may contain some bias, depending on the quality of the data source and the side, in this sense;

- The data are generated by the multiplicity of direct observers, who may not converge in their judgments of the same battle.
- There is no methodological approach to the processing of data or information obtained from conflicts to achieve an overall picture.

Thus, the analysis of historical sources must be increased, for the validation, and improvement of the models, with the direct consequence of improving the quality of the decisions.

3

Automated Support for Battle Operational-Strategic Decision-Making

This chapter proposes an innovative framework that overcomes most limitations of Lanchesterian models and supports decision-making at the highest command levels: the strategic and the operational ones. The framework applies adaptive and predictive control engineering methods to dynamically adjust the prediction to events in the theater of operations and that condition the decision-making. This adaptation mechanism is in itself a learning process in focus on optimization.

Section 3.1 provides an overview of the model. Then Sections 3.2, 3.3, 3.4, and 3.5 describe our framework constituent blocks. Finally, some in-depth calculation procedures are explained in Appendices B and C.

3.1 A Framework to Support Battle Operation-Strategic Decision-Making

There are two principal battle analysis mechanisms alternative to classical Lanchester's models: (i) stochastic models and (ii) deterministic models, some of them in the Lachesterian tradition, e.g., [KMPS17, JHC17a]. Currently, other approaches such as intelligent agents are gaining substantial momentum, e.g., [OT17, ADK17]. These new models aim to extend the capabilities, e.g. [Kre20, Cou19] and reduce the shortcomings of previous approaches, e.g., [Duf17, KLM18]. However, they fail to be an appropriate benchmark for high-level decision-making.

The proposed framework overcomes the limitations of Lanchester's original work, which is profoundly discussed in [Eps85], by treating the battle as a cause-effect process that evolves according to the dynamics of the Lanchester's equations subject to changes and external actions. To do so, the approach applies the adaptive and predictive control theory introduced in [SR95] and incorporates uncertainty modeling techniques. The approach architecture comprises a set of blocks that work cooperatively and ensure that decision-making is carried out coherently, following the military doctrine. In particular, a set of sequential stages trigger the definition of the applicable strategy, the evaluation, and selection of the different possible COAs, and the adaptation of the model to the evolution of the operation. Each block represents the mechanics of military thinking, see Figure 3.1, where $x(t)$ and $y(t)$ define the number of combatants of the x-force and y-force at each instant, $x(t+1)_e$ and $y(t+1)_e$ are the estimated the number of combatants for the following instant.

The implementation requires a logical process capability and should simulate the decision-making process, from prediction to action. In this context, the new framework is formulated and tested in Chapter 4 (it will be robust if its application on real confrontations meets the expectations in terms of performance and consistency).

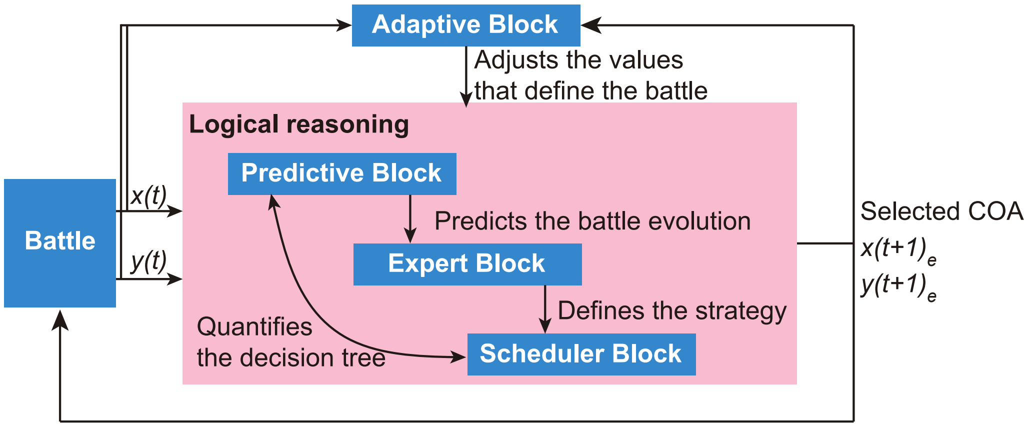

Figure 3.1: The architectural design of our framework. Each block represents the mechanics of military thinking, thus (i) assessing the events of the battle that will define the strategy to be followed and selecting the COA to accomplish the mission, (ii) identifying the resources that will be necessary to carry it out, and finally (iii) adapting to the outcomes.

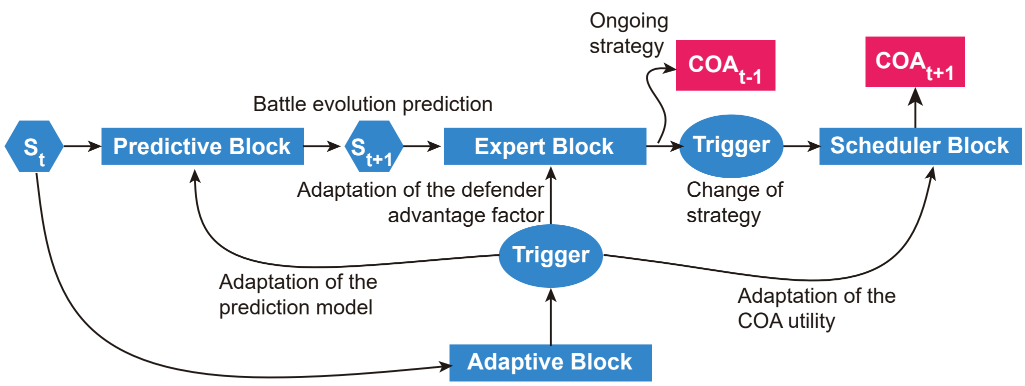

Figure 3.2: Primary elements that trigger the choice of a specific COA in the new framework through a sequential model.

Figure 3.2 develops the essential elements that iteratively trigger the choice of a specific COA. The predictive block generates the predicting evolution. The adaptive block adjusts the parameters of the constituent blocks based on the difference of the output signal (the actual situation) from the predicted one, suitably updated with the last executed COA. The expert block acts trying to modify the trend defined by the predictive block through the scheduler block, thus changing the course of actions following the needs of the battle. It is worth noting that the set-point is related to fulfilling the mission, that the action development times are operation times and, that the available databases with information on conflicts are usually represented by time evolutions in days, in the best case.

3.2 Predictive Block

In military doctrine, intelligence is defined as the interpretation and integration of knowledge about the terrain, meteorology, population, activities, capabilities, and intentions of a present or potential enemy. The intelligence cycle is composed of the phases of direction, acquisition, elaboration, and dissemination. The predictive model will recreate this cycle in the prediction of scenarios necessary to evaluate future decision-making, where the tactics, combat strength, and attrition are identified as the most critical factors for modeling the dynamic prediction of a confrontation. The predictive block defines the future trend of the confrontation at an instant after the current one using Lanchester's equations and a regression model.

3.2.1 Study and conclusions for the practical implementation of the Lanchester combat models.

The Lanchester's equations simplify battle attrition models, emphasizing the importance of troop concentrations in the final outcome. These models were developed during the Great War by F.W. Lanchester [Lan16]. Since then there have been later developments of these laws such as [Dei62] for the mixed law or [Pet67] for the logarithmic law or [Bra95] in his general law.

Literature on Lanchester combat models has grown to provide new insight, thus; [HH95] analyzed the quadratic law using data from the battle of Inchon-Seoul (1950), concluding that the best fit occurred by dividing the battle into sub-battles; [WPY00] studied the stochastic Lanchester form, concluding that other factors play an important role in real combat (strategy, environment, etc.) that the Lanchester's equations do not take into account; [LD04] contrasted daily casualties through data from the battle of Kursk (1943) using Bracken's general Law, concluding the need to divide battles into sequential phases differentiated by major changes in battle concentrations or strategy.

Lanchester's models should be understood as a resource for decision-making of battle dynamics on a local (operational) scale. An explicit approach that seeks to understand, track and anticipate the direct and indirect effects of operational decisions. As the combat models are developed at different levels of abstraction, they can be represented together by Figure 3.3 where the Lanchester models should be positioned as a mechanism for evaluating decisions.

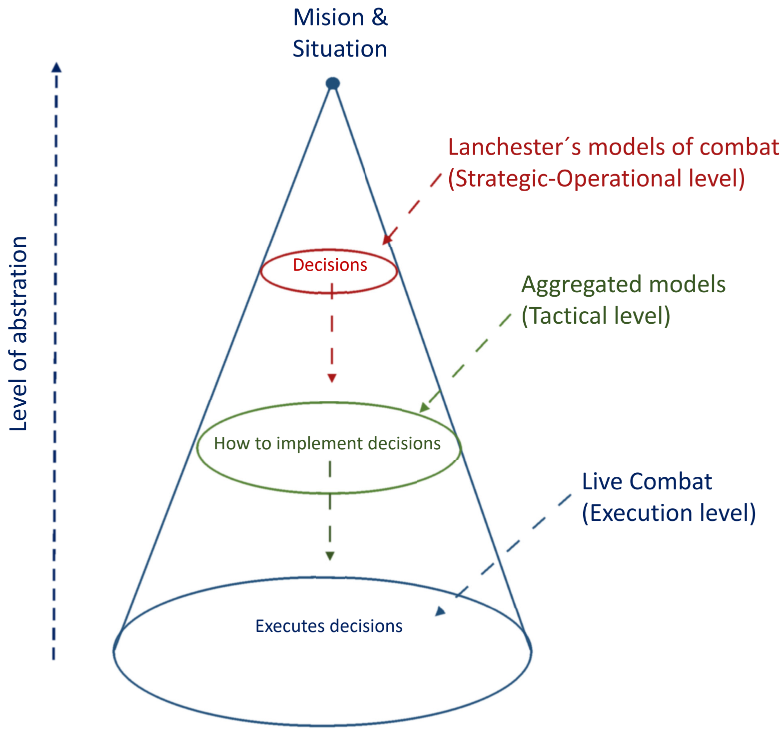

Figure 3.3: The vertical axis identifies the level of abstraction embodied in the model and the base circle of the cone represents reality or complete lack of abstraction, as the level of aggregation increases, the variables defining the level of command gradually abstract details of combat execution. Thus, in the level of application of Lanchester models is in line with levels of strategic-operational aggregation, aggregated models cover up the most basic execution mechanisms of the battle such as individual clashes and the execution level is affected by factors such as weaponry, position, visibility, logistics, etc.

Equations presentation

These equations consider two hostile forces, denoted as x and y. For simplicity, forces are typically modeled as the number of combatants, i.e., the size of each army, although it can be a representation of any element fully engaged in combat with the capacity to generate casualties or losses. Thus, $x(t)$ and $y(t)$ define the number of combatants of the x and y forces at instant t, t is usually measured in days from the beginning of the battle. Additionally, Lanchester's equations consider each force's *lethality*, denoted as a and b, whose calculation depends on the fire, combat typologies, and balance of forces.

- **Case I**, conventional forces in direct-fire combat. Each member of force-x is within range of the enemy and, when force-x takes losses, fire y is concentrated on the remainder, Equations 3.1 and 3.2. Losses will be proportional $ay(t)$ where a is the *lethality* coefficient of y, and equivalently b.

$$\frac{dx}{dt} = -ay(t) \tag{3.1}$$

$$\frac{dy}{dt} = -bx(t) \tag{3.2}$$

We can bring in reinforcements or withdraw troops during the battle, where $f(t)$ and $g(t)$ are the functions that define the reinforcement or withdrawal of troops.

$$\frac{dx}{dt} = -ay(t) \pm f(t) \tag{3.3}$$

$$\frac{dy}{dt} = -bx(t) \pm g(t) \tag{3.4}$$

Considering the case of isolated combats without reinforcements, and solving the system of equations. Equation 3.5 is referred to as Lanchester Square Law.

$$ay(t)_0^2 - ay(t)^2 = bx(t)_0^2 - bx(t)^2 \tag{3.5}$$

Setting $K = ay(t)_0^2 - bx(t)_0^2$ allows depicting the different trajectories of the equation $K = ay(t)^2 - bx(t)^2$. This equation describes a family of hyperbolas in the x-y plane. See Figure 3.4.

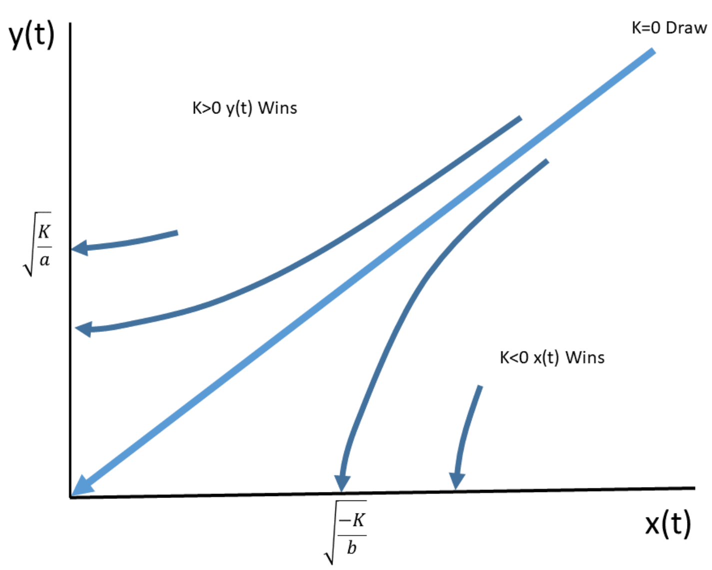

Figure 3.4: Trajectories of Lanchester square law: $K = ay(t)^2 - bx(t)^2$, the arrowheads on the curves represent the trajectory of change of the forces during the fight.

- **Case II**, forces distributed in areas, invisible to the enemy or using concentrated area fires such as artillery, losses are additionally proportional to the number of targets, Equations 3.6 and 3.7;

$$\frac{dx}{dt} = -ax(t)y(t) \tag{3.6}$$

$$\frac{dy}{dt} = -by(t)x(t) \tag{3.7}$$

We can bring in reinforcements or withdraw troops during the battle as the above model.

$$\frac{dx}{dt} = -ax(t)y(t) \pm f(t) \tag{3.8}$$

$$\frac{dy}{dt} = -by(t)x(t) \pm g(t) \tag{3.9}$$

Considering the case of isolated combats without reinforcements, and solving the system of equations. Equation 3.10 is referred to as Lanchester Linear Law.

$$ay(t)_0 - ay(t) = bx(t)_0 - bx(t) \tag{3.10}$$

Setting $K = ay(t)_0 - bx(t)_0$ allows depicting the different trajectories of the equation $K = ay(t) - bx(t)$. This equation describes a family of straight lines in the x-y plane. See Figure 3.5.

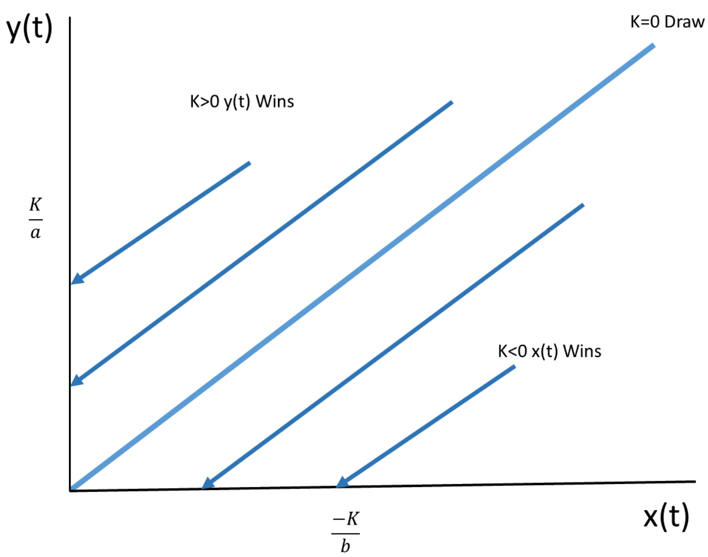

Figure 3.5: Trajectories of Lanchester linear law: $K = ay(t) - bx(t)$, the arrowheads on the lines represent the trajectory of change of the forces during the fight.

- **Case III**, battles between conventional x-forces versus guerrilla-type y-forces (invisible to the enemy). Considering the case of isolated combats without reinforcements, Equations 3.11 and 3.12;

$$\frac{dx}{dt} = -ay(t) \tag{3.11}$$

$$\frac{dy}{dt} = -bx(t)y(t) \tag{3.12}$$

Developing the above cases, we obtain the Lanchester mixed law for asymmetric combat 3.13. This equation describes a family of parabolas in the x-y plane. See Figure 3.6.

$$ay(t) - \frac{1}{2}bx(t)^2 = K \tag{3.13}$$

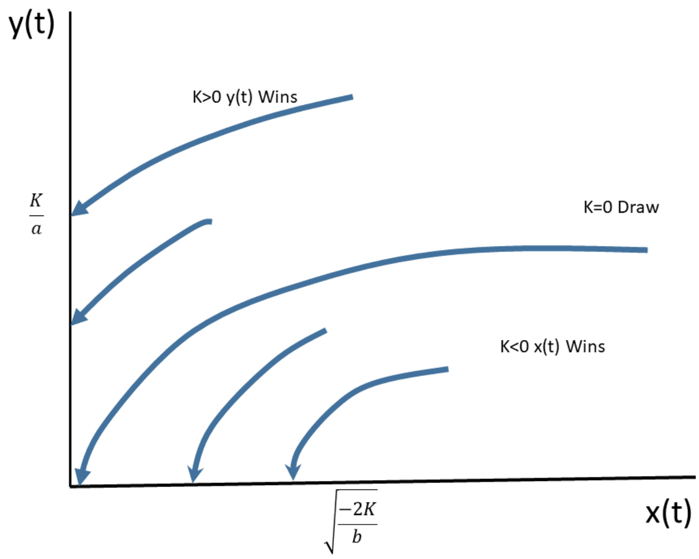

Figure 3.6: Trajectories of Lanchester mixed law: $K = ay(t) - 1/2(bx(t)^2)$, the arrowheads on the curves represent the trajectory of change of the forces during the fight.

- **Case IV**, battles on a big scale between conventional x-forces y-force, the attrition rates are proportional to the number of own troops exposed to fire. Considering the case of isolated combats without reinforcement, Equations 3.14 and 3.15;

$$\frac{dx}{dt} = -ax(t) \tag{3.14}$$

$$\frac{dy}{dt} = -by(t) \tag{3.15}$$

  Developing the above cases, we obtain the Lanchester logarithmic law 3.16.

$$b \ln \frac{x(t)_0}{x(t)} = a \ln \frac{y(t)_0}{y(t)} \tag{3.16}$$

  The Gulf War is a good modern example, where Iraqi casualties were more closely related to the size of Iraqi forces in the combat area than to the size of US forces.

- **Case V**, other representative models treat size differences between x-forces and y-forces as a lethality conditioning factor (inefficiencies of scale [Hel65]), Equations 3.17 and 3.18 for isolated combats. Therefore, Helmbold [Hel65] added E_x and E_y functions that modify the *lethality* of force by a x and y ratio. That is to say, in a very unequal size between the opponents, the larger opponent will not be able to use all its capacity (*Law of diminishing returns* and the smaller one will be more efficient).

$$\frac{dx}{dt} = -ay(t)E_y\left(\frac{x}{y}\right) \tag{3.17}$$

$$\frac{dy}{dt} = -bx(t)E_x\left(\frac{y}{x}\right) \tag{3.18}$$

Where E_y and E_x are inefficiencies of scale factors.

- **Case VI**, there is no reason all types should not be used together. Applying the generalized model defined by [Bra95], it is possible to determine the nature of the battle. Considering the case of isolated combats without reinforcement, Equations 3.19 and 3.20;

$$\frac{dx}{dt} = -a\frac{1}{d}y(t)^p x(t)^q \tag{3.19}$$

$$\frac{dy}{dt} = -bdx(t)^p y(t)^q \tag{3.20}$$

Where d is the tactical parameter that adjusts the *lethality* factor to y-force defender by d or x-force attacker by $1/d$, i.e., it defines which of them has the advantage ($d < 1$ defender advantage, or $d > 1$ attacker advantage, or none when $d = 1$).

Considering the value range of p and q in the interval [0,1], it can be easily related to Lanchester's laws.

  - When $p = 1$ and $q = 1$ the linear law is defined.
  - When $p = 1$ and $q = 0$ the quadratic law is defined.
  - When $p = 0$ and $q = 1$ the logarithmic law is defined.

The lethality coefficient

We can assume that the lethality coefficient is a non-constant coefficient that defines a clear relationship between targets and firer. Other elements can affect the change of an x-force in the course of the battle, such as logistical capabilities, maneuvers, or confusion for example, which should cause variability in the coefficient. Thus, in the case of the square law, the lethality coefficient reflects the degree to which each element of the force can generate losses, in the case of the linear law there is a further dependence on the enemy's force distribution, in the logarithmic law defines losses as proportional to one's forces, even [Hel65] established as conditioning lethality the inefficiencies of scale when the size differences between x forces and y-forces.

Within the lethality coefficient, other elements can favor or diminish it, such as the case of communications [Sch12], defining communication capabilities as an increase in the effectiveness of lethality, or the fatigue factor [RC16], decreasing effectiveness over time in battle.

In the case of battles with a high level of aggregation such as those addressed in this thesis, it is difficult to calculate this coefficient, before the start of the battle, often resorting to statistical calculations, e.g., [Bon67, Bar69], intelligence reports, or expert judgment.

Stochastic Lanchester form

Combat is a very complex process. It is intuited that there are more than force levels to define who wins or loses the battle. It seems worthwhile to explore the stochastic analysis of combat by including random variations in Lanchester models (Lanchester stochastic process). So, applying randomness to the variables representing the strategic and operational command level, among others, includes:

- The initial strength level of the enemy as a random variable.
- Lethality factors as random variables.
- Breakpoints as random variables.
- The casualties timing on each side as a random variable.

Following the path of the last option, where the lethality of each side is assumed to be constant as a simplification, a stochastic Lanchester process can be developed by algorithms as a realization example and contrast.

Developing a Markovian Lanchesterian approximations

Considering that future events only depend on the present state, we can model the evolution of the battle as a continuous Markov chain in which the pattern of confrontation is governed by one of the previous forms, Table A.1. The basis of the chains is the so-called Markov property:

- The changes experienced in the confrontation at time $t+1$ only depend on what happened at time t before.
- With several possible battle states s_1,.....,s_k characterized by the level of force each side in t.
- Where the transitions between states can only occur between neighboring states.

From the Markov property and assuming that the square law rules the fight and the times between casualties, which are determined by an exponential distribution function P where the rate parameter defines the frequency of casualties, a Markovian Lanchesterian model via the inverse transform can be developed, see Figure 3.7.

Thus, Functions 3.21 and 3.22 define the time interval between occurrences of two casualties for x and y force, respectively;

$$\mathrm{P}(\mathrm{t_x} \leq \mathrm{t}) = 1 - e^{-ay(t)t} \tag{3.21}$$

$$\mathrm{P}(\mathrm{t_y} \leq \mathrm{t}) = 1 - e^{-bx(t)t} \tag{3.22}$$

and the inverse transform predicts (calculates) the interval time of casualty for each side from an exponential distribution following a Poisson process, Equations 3.23 and 3.24. Where the random variables u_x and u_y are uniformly distributed on $[0,1]$.

$$\frac{-1}{\mathrm{ay(t)}} \ln(1 - \mathrm{u_x}) = \mathrm{t} \tag{3.23}$$

$$\frac{-1}{\mathrm{bx(t)}} \ln(1 - \mathrm{u_y}) = \mathrm{t} \tag{3.24}$$

Finally, the Markovian Lanchesterian model defined is compared with the deterministic model through the Crete Battle data to explore the differences and draw conclusions, see Figure 3.8.

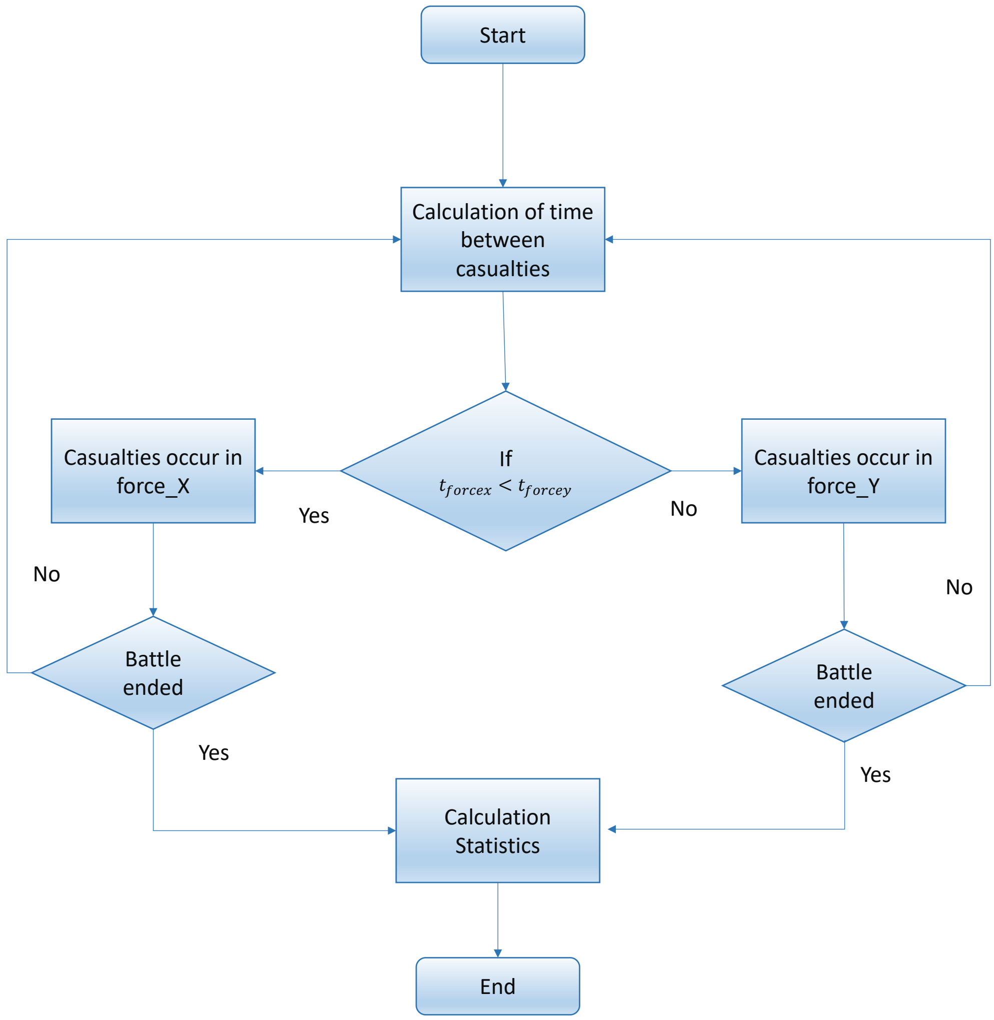

Figure 3.7: The Monte Carlo method generates times interval to the next casualties events of both sides and takes the earlier of two occurrences that trigger the state transition.

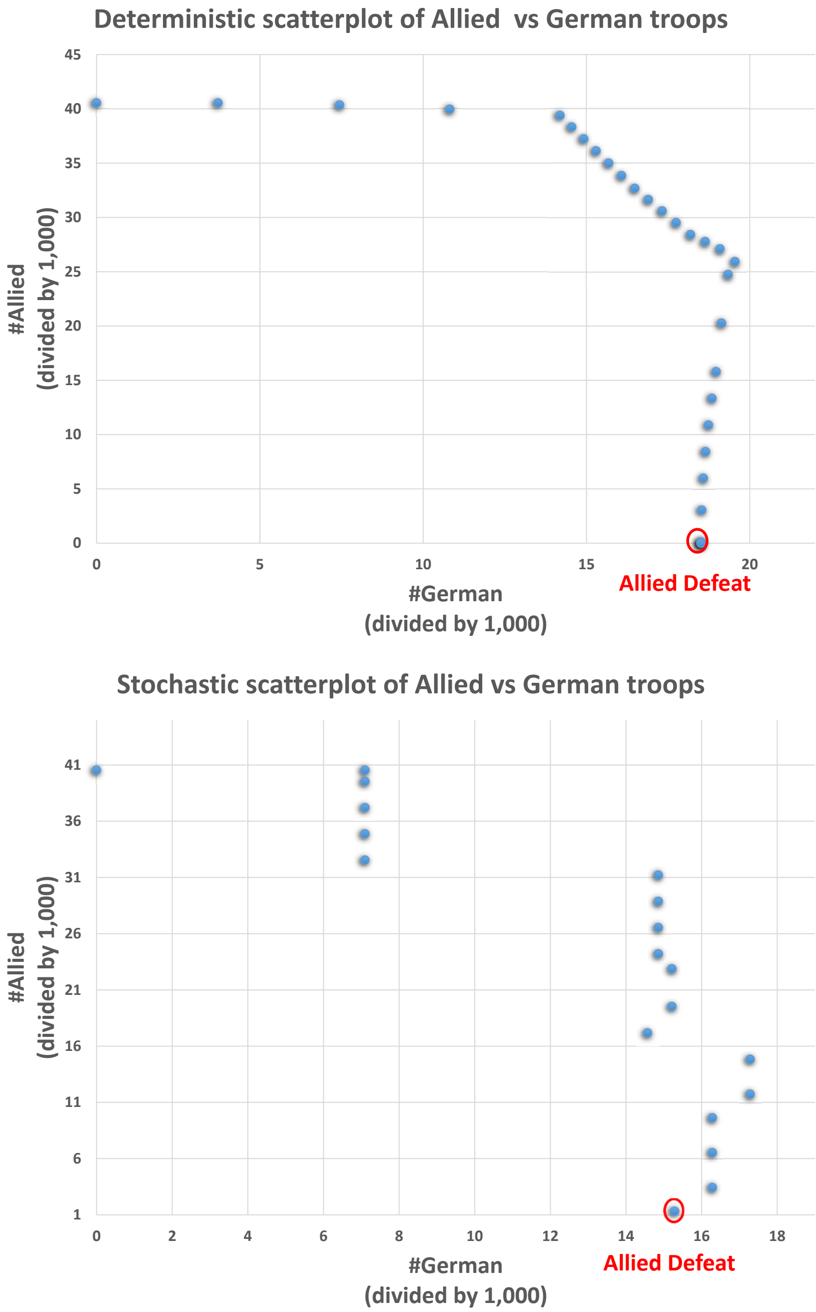

Figure 3.8: In the Crete battle, the deterministic option produces essentially the same results as the stochastic option (German victory), at least in qualitative terms. There is not much to be gained by applying a stochastic model when force levels are large and forces are not close to parity, [Kle80].

Concluding remarks

We have applied the generalized model **Case VI**, in our approach to cover the complexity of today's battles, it should be noted that the tactical parameter d (offensive or defensive strategy) of [Bra95] model is not taken into account because it does not contribute substantially to the adjustment of parameters. After all, battles have different offensive and defensive phases. We have also explored the stochastic Lanchester form, and no qualitative differences were found for the operation-strategic level, where this thesis is aimed.

Furthermore, we have assumed in our approach that the lethality coefficient is a non-constant coefficient dependent on time, events and so as other factors, and that defines a clear relationship between targets and firer.

3.2.2 Generalized Regression Model

Regression attempts to explain the causality of the effects. The generalized model [Bra95] generates four variables to be solved. Using (i) the least-squares method as target function and optimized by the Generalized Reduced Gradient (GRG) algorithm from data obtained during the course of the battle, and (ii) the following metrics that account for the regression model quality: Sum of Squares Regression (SSR), Sum of Squares Total (SST) and R^2, it obtains a feasible estimation procedure to solve the four unknown variables. Therefore, the GRC algorithm manages the slope of the target function as the input values change and determines that it has reached an optimal solution when the partial derivatives are equal to zero.

A higher R^2 value indicates a better fit for the mean daily losses (estimated attrition). A perfect fit would be an R^2 of one.

Coefficient of determination, R^2

The coefficient of determination is a statistic that expresses the proportion of variation explained by the regression, i.e., it is a percentage of how well the variation of one variable explains the variation of the other, is often referred to as R^2.

For each output of a regression procedure (prediction), the residual can be represented as, $y = \hat{y} + e$, where y is the actual value (observation), $\hat{y}$ is the prediction and e is the prediction error. See Figure 3.9.

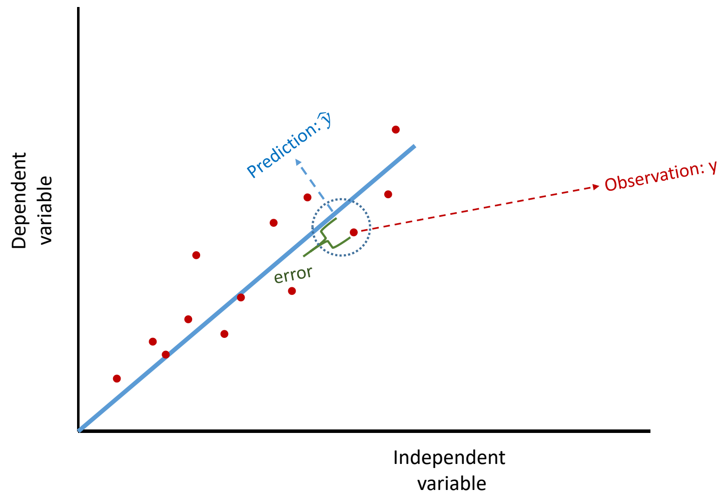

Figure 3.9: Regression is an explanation of causality, between the dependent variable and the independent variables.

From here, we can define the following concepts;

- The explained variation, the sum of squares regression $SSR = \sum(\hat{y}_i - \bar{y}_i)^2$.
- The unexplained variation, the residual sum of squares $RSS = \sum(y_i - \hat{y}_i)^2$.
- The total variation, the sum of squares total $SST = \sum(y_i - \bar{y}_i)^2 = \sum(\hat{y}_i - \bar{y}_i)^2 + \sum(y_i - \hat{y}_i)^2$.
- $R^2 = 1 - \frac{RSS}{SST} = 1 - \frac{\sum(y_i - \hat{y}_i)^2}{\sum(y_i - \bar{y}_i)^2}$; $0 \leq R^2 \leq 1$

Thus, a R^2 value close to 1 identifies a high explanatory power of the regression of the mean daily losses and a value close to 0 identifies a low explanatory power, therefore R^2 is a practical measure that answers the quality of the regression model.

Generalized Reduced Gradient (GRG)

The resolution method used in the prediction of attrition values is nonlinear programming, nonlinear GRG algorithm, whose mathematical structure of which can be analyzed in [Aba78] and [LWJR78], searches for a feasible solution from an initial point and moves in search of the improvement of the Objective function 3.25 in the direction of the solution region that minimizes it, considering the case of isolated combats without reinforcements.

$$Objective function = \frac{1}{n}\sum_{i=1}^{n}(x_{i+1}-(\hat{x}_n-a\frac{1}{d}(\hat{y}_n^p\hat{x}_n^q)h))^2+(y_{n+1}-(\hat{y}_n-bd(\hat{x}_n^p\hat{y}_n^q)h))^2 \quad (3.25)$$

The GRG solving method is suitable for smooth non-linear functions of several variables, but it is conditioned by the chosen initial conditions, so the GRG algorithm must start from a feasible solution. On this issue, it is worth noting:

- A non-linear problem can have more than one solution region, i.e., a set of similar values for the p, q, a, b, where all constraints are satisfied. The different types of battle cataloged, Appendix A, allows the best starting points to be determined.

- The GRG algorithm provides a guided solution from the starting point in the direction of the reduced gradient, following the curvature of the objective function, and satisfying the constraints. Due to a clear dependency between the starting point and the selected trajectory, further iterations of the GRG algorithm could ensure a better solution, simply starting from different starting points. Finally, the solution with the best ratio R^2 - mean squared error (MSE) is selected. See Figure 3.10.

- The GRG algorithm provides a single global solution when the objective function is strictly convex. Otherwise, we will not know if the solution obtained is globally optimal, [SALA98], even sometimes the Solver will stop before finding a locally optimal solution when it is making very slow progress (The cost of the objective function changes very little in the chosen direction).

- The GRG method can find local optima for well-scaled non-convex problems. Due to the precision of computers, when these calculations are performed with very different scaling values from the objective and constraints, the approximation error is accentuated to the extent that the optimal solution cannot be found.

3.3 Expert Block

The development of decision-making is characterized using intelligence resources through the predictive block and its interpretation, leading to the strategy definition. Once the global situation informed by the predictive block has been evaluated, it is necessary to redefine the strategy when there is a change of trend or when such trend change is sought by modifying the strategy (Defensive, Offensive, Stability, etc.). If the previous operational decisions are within the acceptable limits of attrition defined at the set-point, the re-evaluation will not make sense in the first approximation.

3.3.1 Intention Model

The large units, in their advanced movement, make contact progressively. The awareness of the adversary's intentions, in specific areas, allows the selection of an adequate strategy, given the general attitude of the adversary:

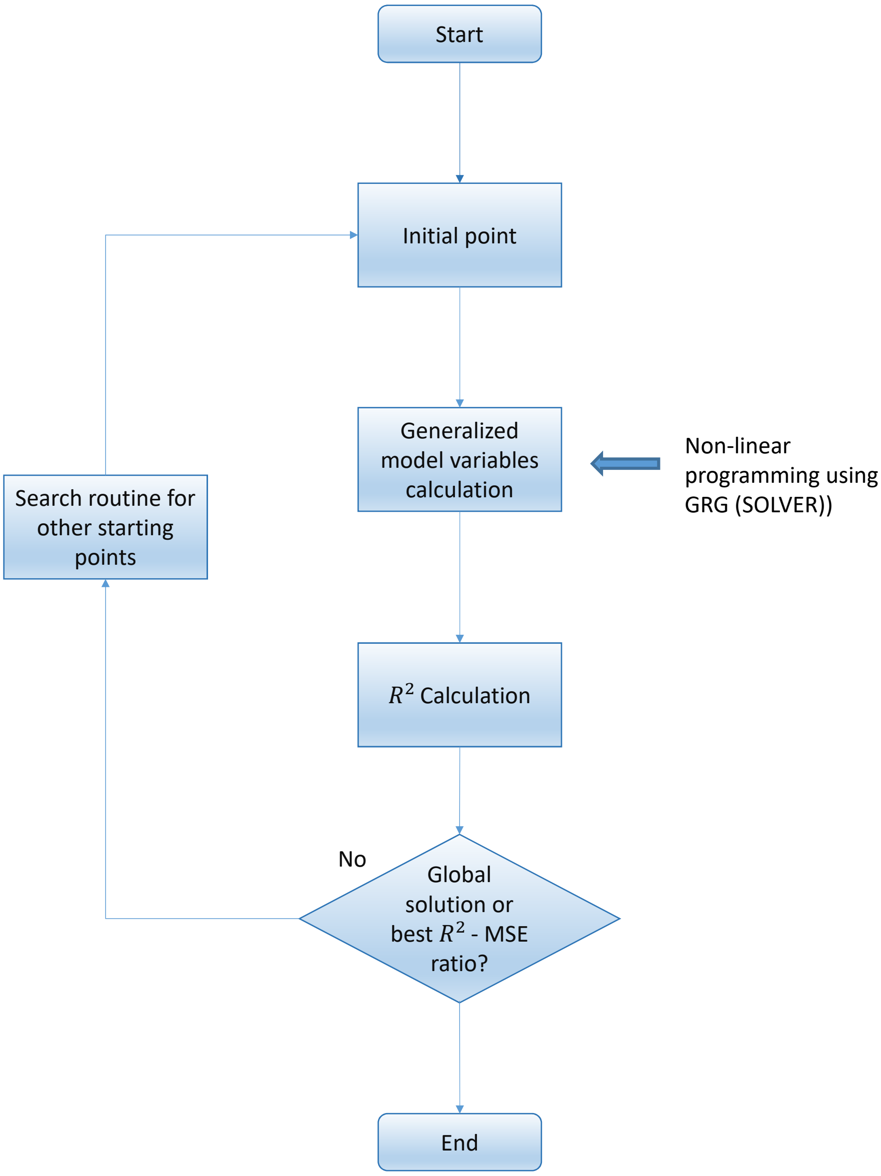

Figure 3.10: The algorithm for calculating the global solution of the generalized model by nonlinear programming using GRG (SOLVER).

- A Defensive Battle means a high risk of being attacked and inferiority of resources.
- An Offensive Battle means a low probability of being rejected and superiority of resources.

The assessment of the adversary's intentions will be based on the actual ability to reject a possible attack in a hostile scenario. The contenders will consider a stable state situation if the probability of a failed attack exceeds the security level. Figure 3.11 depicts a decision tree for evaluating the adversary's intentions.

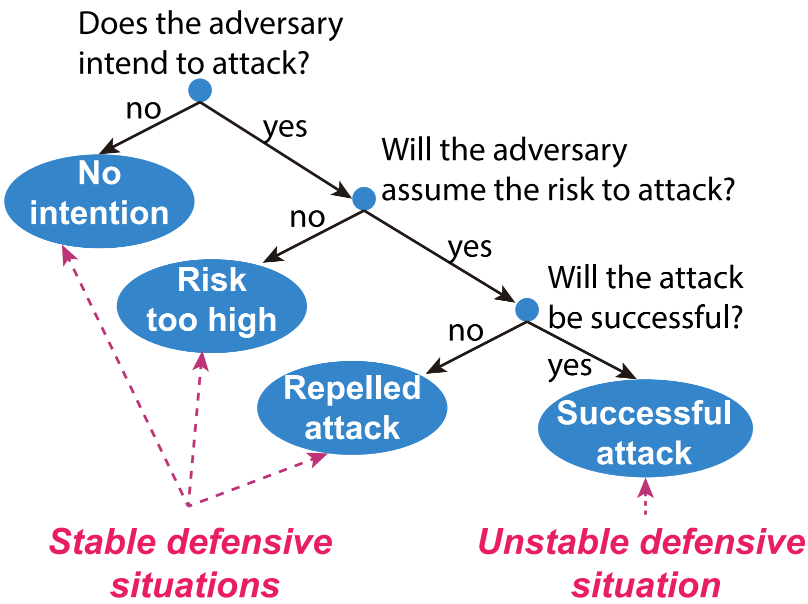

Figure 3.11: Decision tree on adversary intentions in a bipolar situation for assessment, shown in [Chr95] report. If the relationship between one's own forces and the adversary is friendly, the adversary will not consider military aggression. On the other hand, if the relationship is hostile, the adversary may wish to attack, and one's own forces may need a military defense against the adversary.

The intention of an opponent to attack will be given by the minimum probability of success that the opponent needs to launch an attack P (this figure depends on the doctrine of the contender) and by the probability of being rejected by the defender *WinsDef*, Conditions 3.26 and 3.27. Probabilities are defined unilaterally through the opponent's vision, so if the adversary requires a high chance of success of the attack before launching it, the *WinsDef* should be low.

- Equation 3.26 identifies a high risk of being attacked:

$$P < (1 - WinsDef) \tag{3.26}$$

- Equation 3.27 identifies a high risk of being rejected:

$$P > (1 - WinsDef) \tag{3.27}$$

The *WinsDef* curve represents the probability of being rejected by the defender (Equations 3.28 to 3.31), and it is obtained using logistic regression, detailed calculations can be found in Appendix B. This allows estimating the probability of success or failure as a function of the defender's Advantage Factor v as defined in [Hel97], from a subset of data obtained from the CDB90 data set of individual battles, from 1600-1979, available on `https://github.com/jrnold/CDB90` (last visited June 26, 2021). See Figure 3.12.

Lanchester's Square Law defines factor v, where $x(0)$ and $y(0)$ are the numbers of combatants of the x-force attacker and y-force defender at the initial instant, a is the *lethality* of the defender force, and equivalently b of the attacker. Accordingly,

$$\text{WinsDef} = \frac{1}{1 + e^{0.12 - 3.38v}} \tag{3.28}$$

$$v = \ln\sqrt{\frac{\delta}{\alpha}} \tag{3.29}$$

$$\alpha = b\left(\frac{x_0}{y_0}\right) \tag{3.30}$$

$$\delta = a\left(\frac{y_0}{x_0}\right) \tag{3.31}$$

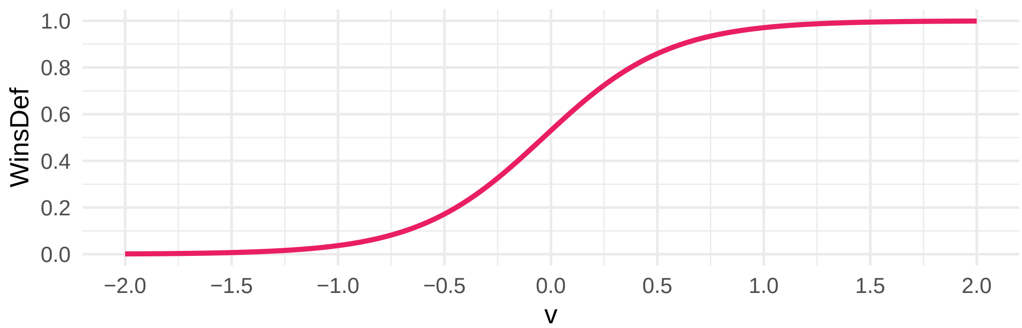

Figure 3.12: Relationship between v and the probability of being rejected by the defender. Empirical evidence shows that the advantage factor favoring the defender has an important influence in determining which side wins, according to Helmbold's work [Hel61].

3.4 Scheduler Block

The COA planning is determined by military doctrine and the different factors of the operational environment, such as, for example, the enemy centers of gravity (COGs). Within the military decision-making process, the planning phase involves COA analysis, comparison, and evaluation, as well as the development of the matrix plan that provides the resources and conditions to optimize and maximize the results.

Action planning is inferred through *decision trees*, which process the doctrinal knowledge (friend and enemy), the strategy defined from the expert block, and evaluate possible outcomes in the context of probable enemy actions obtained through the predictive block.

3.4.1 Alternative Assessment

The assessment of the alternatives is based on the concept of expected value $E(x)$, applicable to random variables that take numerical values, and the utility of the COA. The final objective of the selected COA will be the fulfillment of the mission defined at the set-point. In the current battle decisions, the own casualties x in combat is the main conditioning factor, so the Wald or pessimistic criterion is taken: it is a question of assuring conservative casualties (MAX MIN), Equation 3.32. This criterion involves selecting an alternative whose expected or average attrition is lower.

$$\text{COA}_i = min(E(x)) \tag{3.32}$$

3.4.2 Centers of Gravity

All aspects of planning depend on the determination of well-defined, achievable, and measurable objectives. The process of identifying and defining objectives involves knowing the enemy, geography, and climate of the area of responsibility.

The objective acquisition model will be simplified using the K-Means clustering method (by the tactical disposition of the units in the terrain, using the Euclidean distance as a quantitative variable), obtaining the centers of concentration of the deployed units.

K-Means works by finding clusters with a spherical or convex shape and needs as input data the number of groups in which we are going to segment the population into k cluster, Elbow method, the algorithm according to [BK14], iterates with different values from 1 to n in the sense of reduction of the total sum of intracluster variance. Therefore, for each iteration, it takes the Euclidean distance between each unit with its center and adds up all the squares of the differences calculated (SSE), up to find the *elbow point*, where the SSE vs. cluster curve rate of decline is sharpened. Figure 3.13 shows a practical example of the application of the K-Means plus elbow method algorithm for the determination of Japan Centers of Gravity (COGs) in the battle of Manchuria on 8 August 1945.

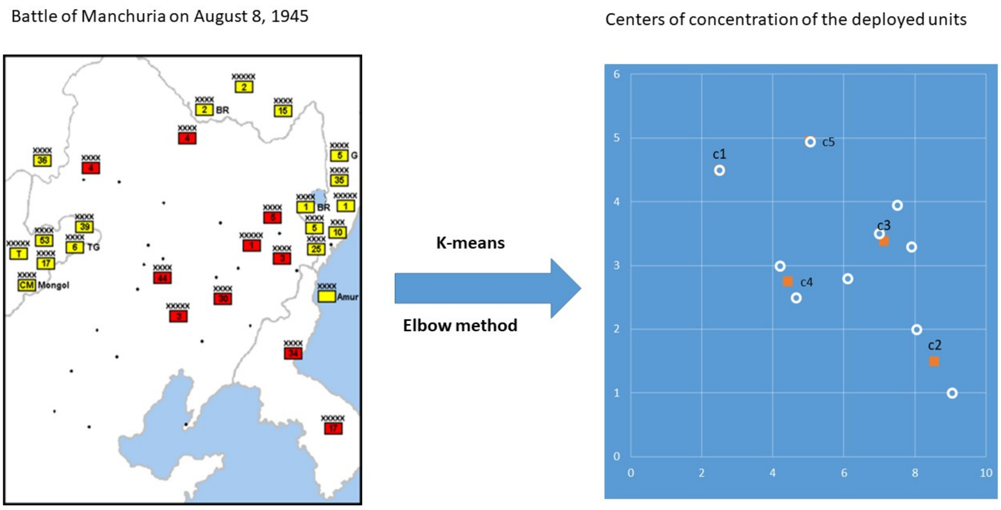

Figure 3.13: The figure on the left depicts the situation described in the biography of the battle of [Gla03] between Japanese (red) and Russian (yellow) forces. The right-hand figure shows the COGs were obtained by applying K-Means plus Elbow method.

3.5 Adaptive Block

Even if a good battle model is available, changes in combat dynamics will lead to the deterioration of the model's fit (prediction and driving). The framework adapts to varying circumstances in the theater of operations and generates changes in the parameters that reflect the decisions' prediction and conditioning. Thus, adaptive control provides a solution theoretically capable of approximating the dynamics of the battle.

The adapting mechanism involves the following tasks:

- Adapting the prediction and factors that determine the strategy to the current battle situation.
- Setting the parameters of the COA usefulness.

This adapting mechanism is a learning process and will provide information for improving the model fit.

3.5.1 Adapting Mechanism

The design of the adapting mechanism has focused on optimizing model prospect (i.e., on error minimization) and improving computational performance.

As Figure 3.14 shows, a *customized auto-tuning control* is used for the predictive block via an approximation to control theory, whose time window is updated step by step with the latest samples. This makes it possible to adapt the values that define the battle to the different phases of the battle, eliminating the jumps produced by random errors or outliers.

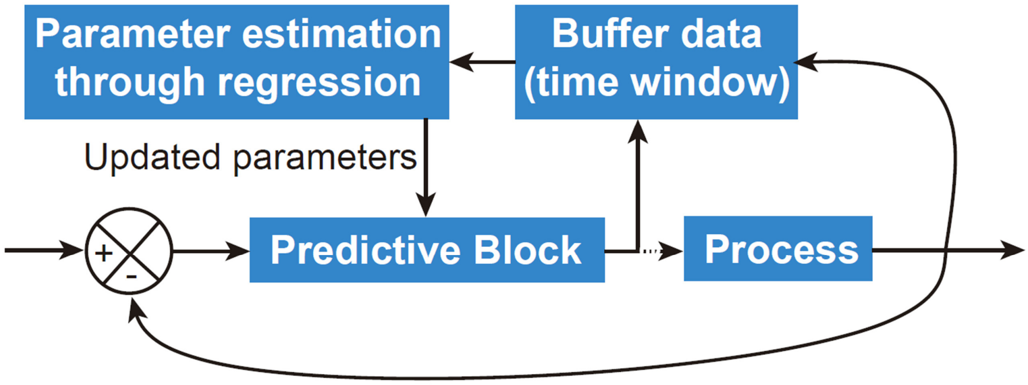

Figure 3.14: The adaptive auto-tuning control approximation recursively estimates the parameter values of the predictive model. The most important aspect of this type of control is having a sufficiently robust technique of parameter estimation.

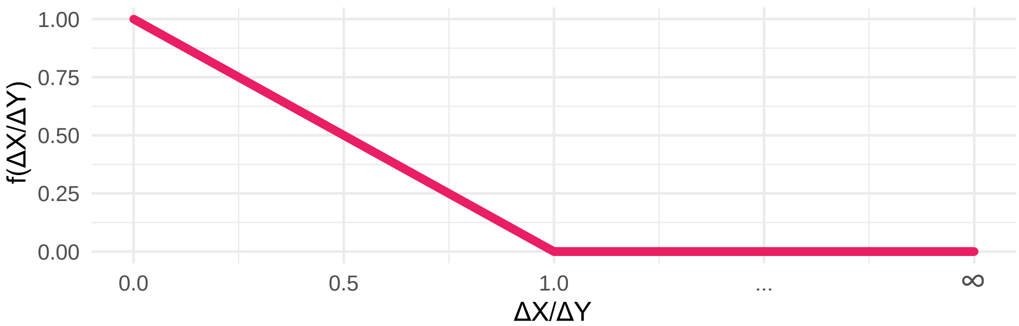

Figure 3.15: *The utility* is a function that relates casualties among opponents $U_i = f(\triangle x / \triangle y)$, where $\triangle x$ stands for own casualties and $\triangle y$ for enemy casualties. It is worth noting that the utility function is close to 1 for COAs that maximize enemy casualties and minimize their own.

A supervised learning mechanism is used for the expert block adaptation, which extends the binary values (final result) of previous battles and recalculates the logistic regression base of the intention model according to the advantage factor. Adaptation is carried out after the final outcome.

Finally, as Figure 3.15 shows, in the case of the scheduler block, a utility function is used as an adaptation measure to represent the effectiveness in taking planned actions (Friendly Options) by casualty ratio. Feeding the effectiveness of the previously selected COAs concerning the opponent's actions, it will provide the new framework approach with a discarding capability for future tree constructions, avoiding the selection of inefficient COAs.

4 Empirical Validation

In order to answer RQ2, this chapter reports the empirical validation of the framework on three of the greatest battles of the Second World War: Crete (Section 4.1), Iwo Jima (Section 4.2), and Kursk (Section 4.3). In these battles, the type of combat was mainly land-based. As this mode of combat has not essentially changed since then, the experimental results should be extrapolated adequately to present-day combat.

In particular, on the Battle of Crete and Iwo Jima, our validation goal is to identify the best possible courses of action according to current doctrine and determine the effects they produce on the adversary in comparison with the actual battles on 20 May 1941 and 19 February 1945, on the Kursk battle, our goal is the correct identification of the battle phases by dynamically adjust from adaptive and predictive control, Figure 4.1.

The initialization values of the validation cases can be found in Appendix D.

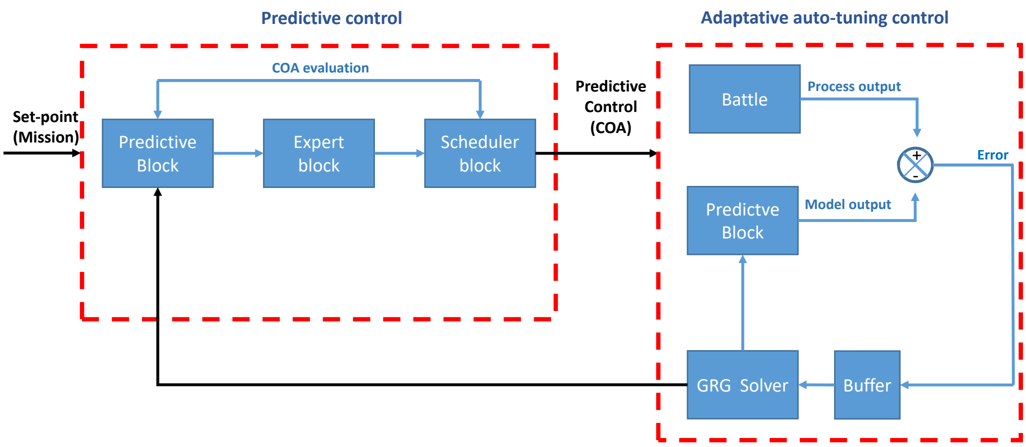

Figure 4.1: The basic adaptive predictive control scheme for practical application. The adaptive control mechanism makes the discrepancy between the battle process output and the predictive model outputs tend to zero, highlighting the dual role played by the predictive block in the system at each sampling time window.

4.1 The battle of Crete

4.1.1 Historical overview

On 27 April 1941, A. Hitler ordered to invade the island of Crete. Airborne troops carried out the operation under the command of General K. Student on 20 May, involving 700 transport planes and 750 gliders supported by the Luftwaffe. The island's invasion was undertaken by 22,000 German paratroopers and mountain troops, and 2700 Italian troops, who took less than two weeks to occupy it. The Allies had 42,547 men of different nationalities. The British evacuated their positions, protected by the Royal Navy, which suffered heavy losses. Crete remained in German hands until its garrison capitulated in May 1945. Crete remained in German hands until its garrison capitulated in May 1945. See Figure 4.2.

Figure 4.2: Crete's invasion on 20 May 1941 provided by Made in Crete.

According to the historical data:

- 8100 German paratroopers landed on the first day, 7400 on the second day, and 9500 more evenly over the following days.
- In the different areas of Crete, there were deployed: 27,550 British Empire soldiers, 13,000 Greek soldiers, as well as unarmed 4000 to 5000 Cypriots and Palestinians.
- A total of 950 British soldiers landed on the eighth day of the battle.
- Two Greek battalions left the battle when their armament and ammunition ran out, evenly from the third day of the fight. Another 2800 Greek soldiers were captured or killed.

- Approximately 4000 British troops were evacuated on the tenth day of the fighting, and another 11,000 evenly through the thirteenth day of the battle, and 1000 more on the thirteenth day.
- Germans estimated their casualties at 6000, while the British estimated 9000 Germans wounded and 6000 killed.
- There were 2600 British and 2600 Greek soldiers dead. Additionally, 10,500 British and 5600 Greek soldiers were captured.

4.1.2 Battle Analysis

This section describes the dataset and then summarizes the analysis the framework provides.

Dataset

Table 4.1 and Figure 4.3 describe the dataset regarding the landing of German troops and the landing or withdrawal of Allied troops during the invasion of Crete and subsequent evolution. This dataset was obtained from the combined study of the following literature sources: [Eng63], [Bia14], [Cox01], and [Mil89]. Additionally, it is assumed that Lanchester's Square Law is fulfilled for the acquisition of intermediate casualty data not available in the literature. Please note that numbers are divided by 1000. For example, the first row in Table 4.1 shows that at the beginning of the first day, 40,550 Allied troop soldiers and 8100 German paratroopers landed at Crete in a naval manner. A negative value in the 'departure' column means that new troops landed, e.g., at the beginning of the 7th day, there were a total of 18,187 German and 28,431 Allied soldiers on the island, and 1357 additional paratroopers and 600 new Allied soldiers landed.

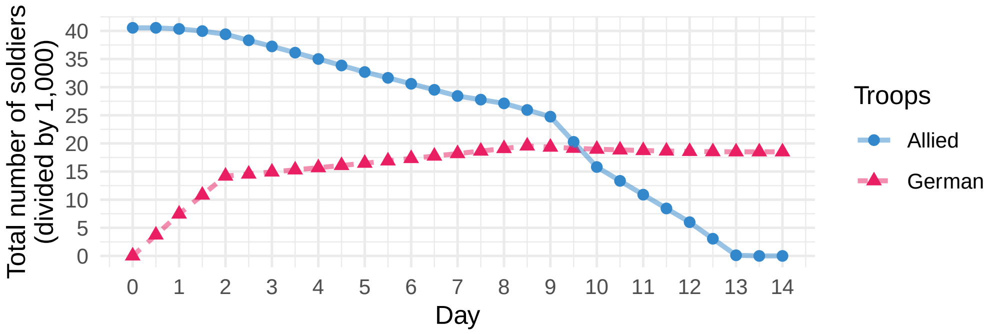

Figure 4.3: German and Allied troops evolution. As the German troops managed to transport enough units to defeat the garrison, Allied troops progressively lost the battle. Although the invasion was successful, there were heavy casualties among the German troops.

Predictive Block

Given the aggregated values in Table 4.1, the predictive block defines which parameter values (p, q, a, and b) fit best the data, using a generalized version of the Lanchester's equations 3.19 and 3.20 provided by [Bra95].

In this dynamic process, parameter values are adjusted by the adaptive block, step by step through the battle's evolution, according to the selected time window evolving incrementally with the latest samples (7 sample-equivalent to 3 battle days). The procedure used determines the parameter values, applying the *generalized regression model* depicted in Sections 3.2.2 and 3.5.

	Total Number		Instant Change	
Days	G. Troops	A. Troops	G. Paratroopers	A. Toops Departure
0	0	40.55	8.1	0
0.5	3.721545	40.55	0	0
1	7.44309	40.35647966	7.4	0
1.5	10.81620251	39.96943898	0	0
2	14.19245006	39.40699645	1.357	0.667
2.5	14.55175339	38.33548905	0	0
3	14.91973593	37.24529787	1.357	0.666
3.5	15.29654901	36.1364716	0	0
4	15.68234359	35.00805105	1.357	0.667
4.5	16.07727838	33.85906919	0	0
5	16.48151992	32.68955071	1.357	0.35
5.5	16.89523456	31.65751167	0	0
6	17.31730871	30.60395948	1.357	0.35
6.5	17.74791664	29.52845942	0	0
7	18.18723612	28.43056776	1.357	-0.6
7.5	18.63544852	27.78483148	0	0
8	19.08889139	27.11578816	1.357	0.35
8.5	19.5477535	25.94816581	0	0
9	19.33757336	24.75668262	0	6.95
9.5	19.13704423	20.27612881	0	0
10	18.97280759	15.80600251	0	2.95
10.5	18.84477897	13.34441651	0	0
11	18.73668919	10.88948801	0	2.95
11.5	18.64848434	8.440180169	0	0
12	18.58011888	5.995458984	0	3.95
12.5	18.53155566	3.054292802	0	0
13	18.50681589	0.115651907	0	0
13.5	18.50587911	0	0	0
14	18.50587911	0	0	0

Table 4.1: Crete battle dataset gathered from [Eng63], [Bia14], [Cox01] and [Mil89]. Data imputation was performed assuming Lanchester's Square Law. Decimal notation is used, dividing actual numbers by 1,000. G and A stand for German and Allied.

Figure 4.4 shows how the values fit around the target values that define the entire battle shown in Table 4.2 at each iteration, obtaining values of R^2 close to 1, detailed values are available[1]. This demonstrates that the treatment by Lanchester's equations of major battles must be done in phases due to changes in the troop *lethality*, as well as the variation in the typology of the confrontations and armaments used, agreeing with other authors as [LD04], [RC16] and [Che03].

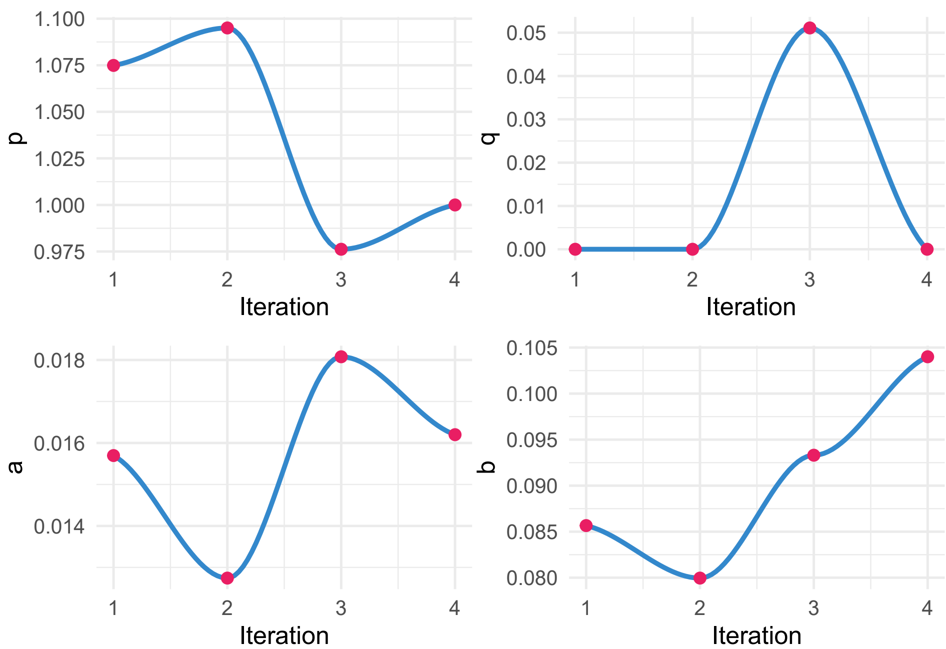

Figure 4.4: Parameter values evolution by adaptive block. Fitted values roughly coincide with the actual ones in dataset, Table 4.2.

[1] https://github.com/gminguela/the-empirical-validation-of-the-framework-on-the-Battle-of-Crete-Iwo-Jima-and-Kursk-and-others

Parameter	Calculated Values
p	1
q	0
a	0.0162
b	0.104

Table 4.2: Target parameter values obtained from Engel [Eng63] (Data source on https://apps.dtic.mil/sti/citations/AD0298786).

Expert Block

The assessment of German Troop intentions, following the procedure depicted in Section 3.3.1, identifies a high risk of attack on Allied troops, the probability of success P that German troops need to launch an attack is much lower than that obtained in the assessment, Equation 4.1. See Figure 4.5.

$$P_{Germantroops} < (1 - WinsDef) \tag{4.1}$$

The German Troop intentions allow the selection of an adequate strategy for the Allied troops. In this case, a *Defensive strategy* is chosen due to the following principles:

- *Principle of concentration.* The side of the opponent with the greater strength, all other factors being equal, will inflict the greater damage.
- *Law of the casualty distribution.* The opponent with greater strength will be the one that receives fewer casualties.

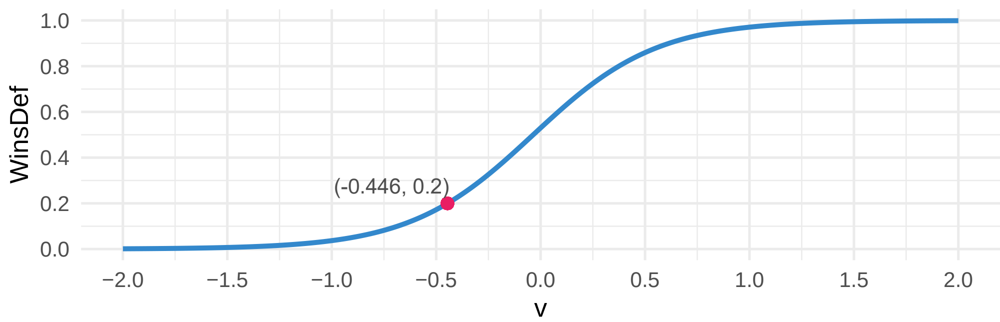

Figure 4.5: Assessment of the adversary's intentions. The plot shows the Allied troops' deficient capability to reject attack (20%), with a defender's Advantage Factor $v = -0.446$. Therefore, $P_{Germantroops} < 80\%$, matching with the real intention of the German Strategy on 27 April 1941.

This assessment took into account the following points:

- If the adversary needs a high probability of success to launch the attack, the adversary has a high risk of aversion (otherwise, a low risk of aversion). Since opponents are unaware of their enemy's aversion risk, this parameter is estimated.
- The risk of aversion is conditioned by the sizes of the armies and the uncertainty of the available information. In our case, the risk of aversion of German troops is low.

Scheduler Block

The evaluation of the alternative COA that could have been carried out to prevent the defeat of the Allies will be performed using decision trees, following the procedure depicted in Section 3.4.1, which considers the possible battle outcomes obtained from the predictive block.

Before the evaluation, the following should be taken into account: German troops occupied Crete island without numerical superiority, the effectiveness factor of the German troops was the cause for the Allied defeat as German paratroopers and mountain troops were better trained, motivated, and organized, as opposed to Allied troops, which were poorly equipped, worn out, poorly trained and organized by nationalities. From the above conclusions and taking into account the strategy defined in the previous Section 4.1.2, we will define a series of operational options that would avoid the defeat.

- *Increasing lethality:* Greek troops were poorly armed. This course is selected to increase the factor of *lethality* by supplying armaments.
- *Defensive position, fortified terrain*: The Allies were not prepared for the defense of the island, the maneuver of work, and the creation of fortified zones that would have prevented the island invasion.

This evaluation took into account the following points:

- In the case of a frontal attack, the ability to reject it is conditioned by Law 3:1 of land combat, i.e., the defender has an advantage factor of 3 to 1 whenever it is deployed on favorable terrain and with a defensive position. In other situations, the most appropriate ratio is 1.5:1, according to the research conducted by [Dav95]. Thus, applied to the evaluation, it means the increase of the allied force by a factor of 1.5.
- According to [Str11], in most historical combats, the relations between initial and final concentrations of forces are relatively high, considering that the breakpoint of the battle (the end) takes place with attrition of forces > 30% to a contender.

After the assessment, see Figure 4.6 where the different COAs are developed, Figure 4.7 where the scatter plot for the winning option is depicted, and Figure 4.8 where the scatter plot for the defeated option is depicted, the chosen COA is *defensive position, fortified terrain* concerning assuring conservative casualties and avoiding the occupation of Crete.

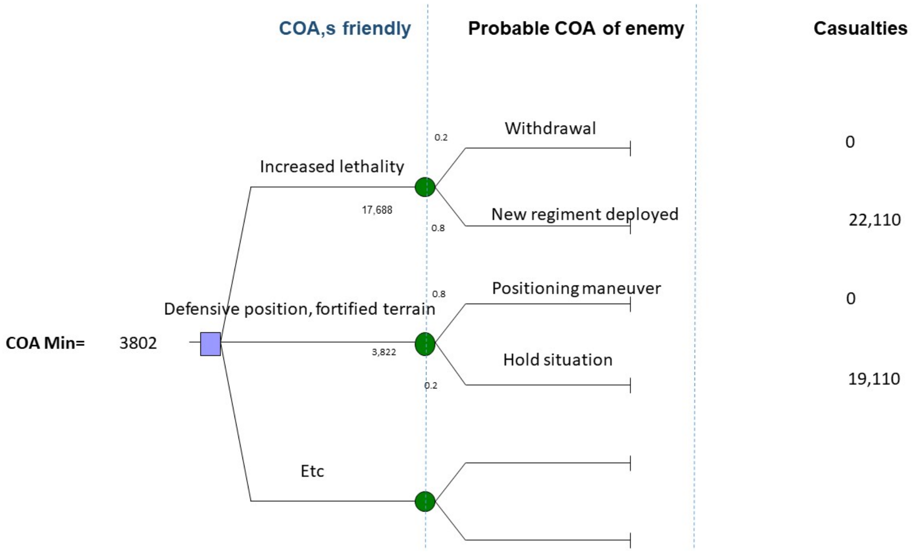

Figure 4.6: Development of the various courses of action, each Allie COA creates likely reaction alternatives for the German strategy, quantified through its doctrine. The final expected value for each COA defines the best choice.

4.2 The battle of Iwo Jima

4.2.1 Historical overview

The Battle of Iwo Jima, which began on February 19, 1945, is part of the Pacific War, in which the United States Marine Corps and Navy landed and eventually captured the island of Iwo Jima from the Japanese Army in the

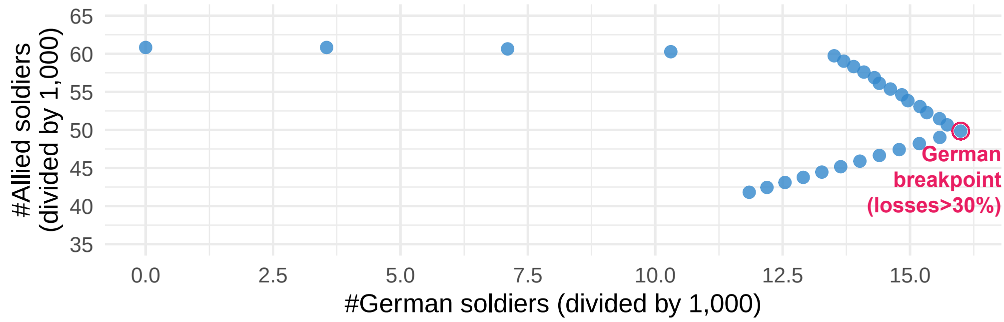

Figure 4.7: Scatter plot obtained from the predictive block for *defensive position and fortified terrain* as an alternative course by applying a 1.5:1 ratio for Allied troops. The selected COA prevents the invasion of the island (the German breakpoint on the 8th day defines the Allied victory).

final throes of the Second World War. The US invasion, called Operation Detachment, was targeted at capturing the island with its two airfields: South Field and Central Field. The Japanese Army positions were heavily fortified, with a dense network of bunkers, hidden artillery positions, and tunnels. The US ground forces were supported by naval artillery and air support throughout the battle. The five-week US siege caused the casualties of almost the entire Japanese force. See Figure 4.9.

According to the historical data:

- The battle lasted 36 days.
- Three landings take place on D-Day, D+2, and D+5 with a sequence of 54,000, 6,000, and 13,000 soldiers.
- The approximate number of Japanese troops on the island was 21,500.
- The number of active American casualties during the land combat was approximately 22,000 troops and the total Japanese forces.

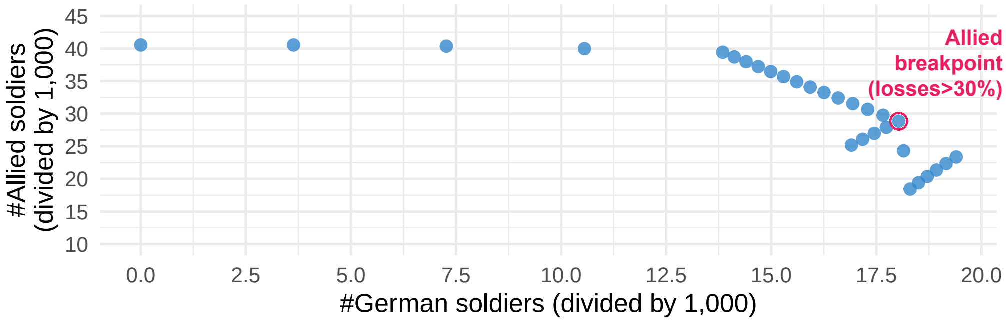

Figure 4.8: Scatter plot obtained from the predictive block for *increasing lethality factor* as an alternative COA, applying an increased lethality factor of 25%. This shows how the other COA does not prevent the invasion of the island (the Allied breakpoint on the 8th day defines the Allied defeat).

4.2.2 Battle Analysis

This section describes the battle dataset and then summarizes the analysis the framework provides.

Dataset

Table 4.3 and Figure 4.10 describe the dataset regarding the landing of US troops during the invasion of Iwo Jima and subsequent evolution. This dataset was obtained from the combined study of the following literature sources: [Eng54], [Ale94], and [Mor46]. Additionally, it is assumed that Lanchester's Square Law is fulfilled for the acquisition of intermediate casualty data not available in the literature. Please note that numbers are divided by 1000.

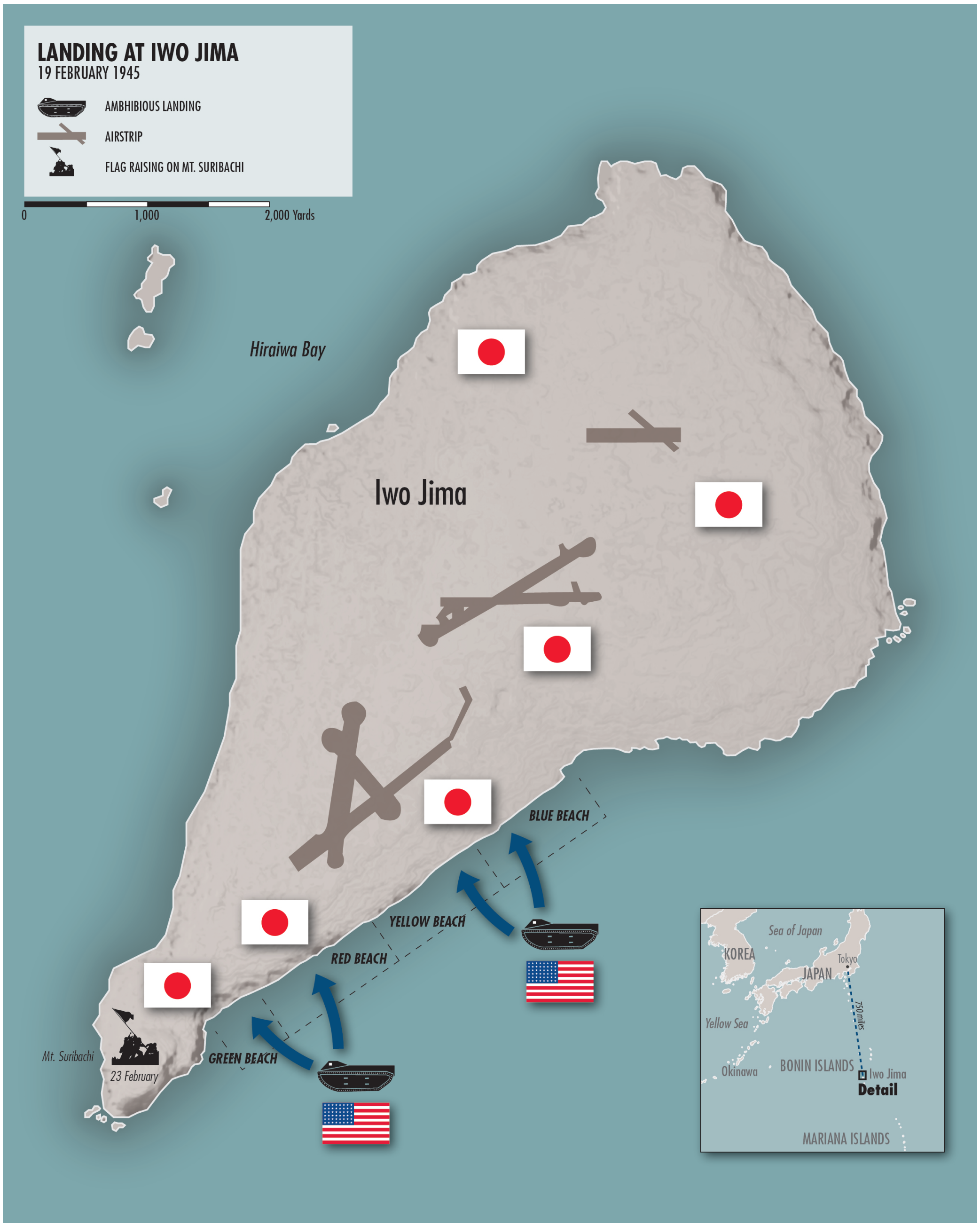

Figure 4.9: Iwo Jima landing plan provided by National World War II Museum in New Orleans.

	Total Number		Instant Change	
Days	U. troops	J. troops	U. Marines landed	J. troops departure
0	0	21.5	0	0
0.5	0	21.5	0	0
1	0	21.5	27	0
1.5	26.40445	21.5	27	0
2	52.8089	21.36005642	0	0
2.5	52.21722644	21.08016925	3	0
3	54.63330575	20.80341794	3	0
3.5	57.05705107	20.51386142	0	0
4	56.48881711	20.21145905	0	0
4.5	55.92895969	19.91206832	6.5	0
5	61.8773954	19.61564484	6.5	0
5.5	67.83404204	19.28769464	0	0
6	67.2997729	18.92817422	0	0
6.5	66.77546247	18.57148542	0	0
7	66.26103233	18.21757547	0	0
7.5	65.75640549	17.866392	0	0
8	65.26150643	17.51788305	0	0
8.5	64.77626107	17.17199707	0	0
9	64.30059675	16.82868288	0	0
9.5	63.83444223	16.48788972	0	0
10	63.37772769	16.14956718	0	0
10.5	62.93038468	15.81366522	0	0
11	62.49234615	15.48013418	0	0
11.5	62.06354643	15.14892475	0	0
12	61.64392122	14.81998795	0	0
12.5	61.23340755	14.49327517	0	0
13	60.83194383	14.16873811	0	0
13.5	60.43946978	13.84632881	0	0
14	60.05592648	13.52599962	0	0
14.5	59.68125629	13.2077032	0	0

	Total Number		Instant Change	
Days	U. troops	J. troops	U. Marines landed	J. troops departure
15	59.31540291	12.89139255	0	0
15.5	58.95831133	12.57702091	0	0
16	58.60992786	12.26454186	0	0
16.5	58.27020005	11.95390924	0	0
17	57.93907676	11.64507718	0	0
17.5	57.61650812	11.33800008	0	0
18	57.30244552	11.03263258	0	0
18.5	56.9968416	10.72892962	0	0
19	56.69965025	10.42684636	0	0
19.5	56.4108266	10.12633822	0	0
20	56.13032703	9.827360834	0	0
20.5	55.85810914	9.529870101	0	0
21	55.59413174	9.233822123	0	0
21.5	55.33835486	8.939173224	0	0
22	55.09073977	8.645879944	0	0
22.5	54.85124889	8.353899023	0	0
23	54.61984589	8.063187404	0	0
23.5	54.3964956	7.77370222	0	0
24	54.18116405	7.485400794	0	0
24.5	53.97381844	7.198240624	0	0
25	53.77442718	6.912179387	0	0
25.5	53.58295981	6.627174923	0	0
26	53.39938706	6.343185236	0	0
26.5	53.22368083	6.060168484	0	0
27	53.05581417	5.778082976	0	0
27.5	52.89576127	5.496887161	0	0
28	52.74349749	5.216539626	0	0
28.5	52.59899935	4.936999089	0	0
29	52.46224447	4.658224393	0	0

	Total Number		Instant Change	
Days	U. troops	J. troops	U. Marines landed	J. troops departure
29.5	52.33321166	4.380174497	0	0
30	52.21188082	4.102808475	0	0
30.5	52.09823303	3.826085507	0	0
31	51.99225046	3.549964872	0	0
31.5	51.89391643	3.274405944	0	0
32	51.80321539	2.999368187	0	0
32.5	51.72013289	2.724811146	0	0
33	51.64465562	2.450694441	0	0
33.5	51.57677138	2.176977767	0	0
34	51.5164691	1.903620878	0	0
34.5	51.4637388	1.630583592	0	0
35	51.41857164	1.357825776	0	0
35.5	51.38095986	1.085307347	0	0
36	51.35089685	0.81298826	0	0
36.5	51.32837707	0.540828506	0	0

Table 4.3: Iwo Jima Battle dataset gathered from [Eng54], [Ale94], and [Mor46]. Data imputation was performed assuming Lanchester's Square Law. Decimal notation is used, dividing actual numbers by 1,000. U and J stand for US and Japan. Fatigue-related medical casualties and non-combat support personnel have not been considered.

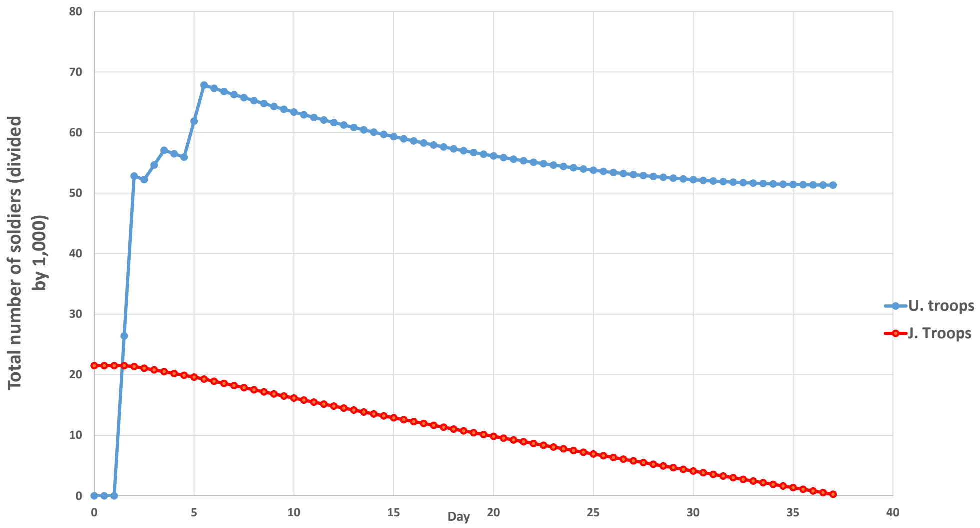

Figure 4.10: Evolution of Japanese and US troops. As the US troops managed to land enough units, the Japanese troops progressively lost the battle. The Japanese army positions were heavily fortified, resulting in heavy casualties among the Marine Corps before the Japanese garrison was defeated.

Predictive Block

Given the aggregated values in Table 4.3, the predictive block defines the parameter values (p, q, a, and b) that best fit the dataset and according to the selected time window evolving incrementally with the latest samples, following the procedure described in Sections 3.2.2 and 3.5.

Figure 4.11 shows how the values fit around the target values that define the entire battle shown in Table 4.4 at each iteration, defining phases due to changes in the troop *lethality* and obtaining values of R^2 close to 1, detailed values are available[2].

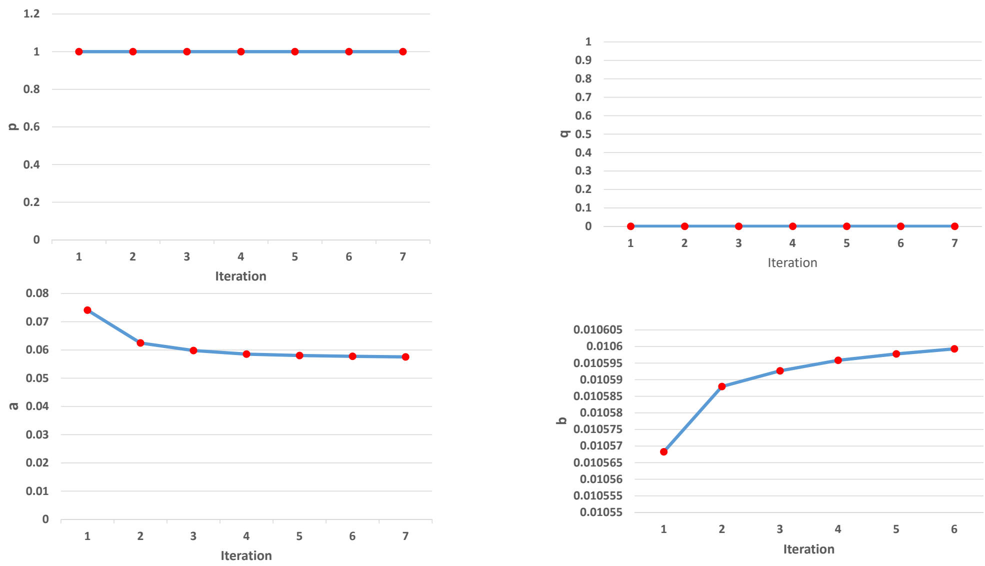

Figure 4.11: Parameter values evolution by adaptive block. Fitted values roughly coincide with the actual ones in dataset, Table 4.4. In this analysis, we can see an increase in the *lethality* of Japanese troops (a coefficient) in the 1 iteration compared to the [Eng54] work, coinciding with the Marine Corps landing phase, where the Marines were most at risk. This as well as previous analyses demonstrate that the treatment by Lanchester's equations of major battles must be done in phases due to changes in the troop *lethality* and the type of combat.

Expert Block

Following the procedure described in Section 3.3.1 identifies the probability of success P through the defender's Advantage Factor, Figure 4.12, which allows the selection of an appropriate strategy for Japan's troops. In

[2] https://github.com/gminguela/the-empirical-validation-of-the-framework-on-the-Battle-of-Crete-Iwo-Jima-and-Kursk-and-others

Parameter	Calculated Values
p	1
q	0
a	0.0554
b	0.0106

Table 4.4: Target parameter values obtained from [Eng54] (Data source on URL: https://www.jstor.org/stable/166602).

this case, a *Defensive strategy* is chosen because the probability of success that US troops will need to launch an attack is much lower than that obtained in the assessment (see Equation 4.2) and based on the provision of a large, well-equipped, and well-supported US force in the area of operations, following the *Principle of concentration* and the *Law of the casualty distribution* in this analysis.

$$P_{UStroops} < (1 - WinsDef) \tag{4.2}$$

Moreover, the US aversion risk to the combat was low what conditions the likelihood of US success to launch the attack, as in the previous case of Crete Battle, based on the need to avoid detection of the B-29 bombers in their raid towards Japan, a key point in the global strategy.

Scheduler Block

The evaluation of the alternative COA that could have been carried out to prevent the invasion or minimize the final casualties follows the procedure depicted in Section 3.4.1.

Before the assessment, the following points should be addressed: Amer-

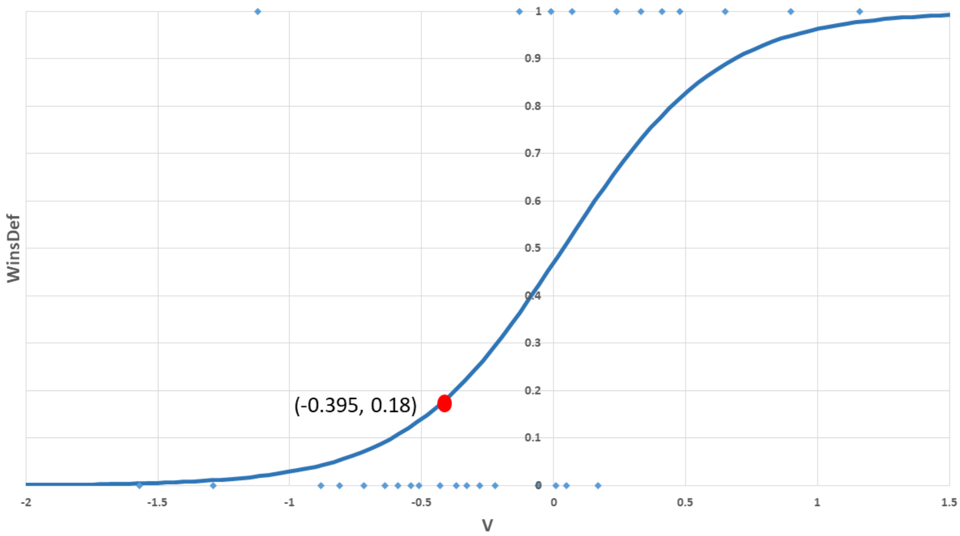

Figure 4.12: Assessment of the adversary's intentions. The plot shows the Japanese troops' deficient capability to reject attack (18%), with a defender's Advantage Factor $v = -0.395$. Therefore, $P_{USAtroops} < 82\%$, matching with the real intention of the US Strategy on 19 February 1945.

ican troops occupied the island of Iwo Jima with numerical superiority and the *lethality* factor of American troops was not the main cause of Japan's defeat, rather inadequate logistical support in terms of ammunition, shells, food, reinforcements, and water.

From the above conclusions and taking into account the strategy defined in the previous Section 4.2.2, we will define a series of operational options.

- *Reinforcement*: Japan troops were defensive position and fortified terrain but very poor logistical operation. This course is selected to increase the reinforcement that would have prevented the island invasion.

- *Withdrawal*: Withdrawal or surrender in the face of possible lack of external support, Japan's WWII strategy based on the last imperial soldier, makes no sense in today's military doctrines.

This evaluation took into account the following rules; Law 3:1, the 30% breakpoint, and the current doctrine.

After the assessment, see Figure 4.13 where the different COAs are developed, Figure 4.14 where the scatter plot for the only available win option is plotted, and Figure 4.15 where the scatter plot for the other interesting option is plotted.

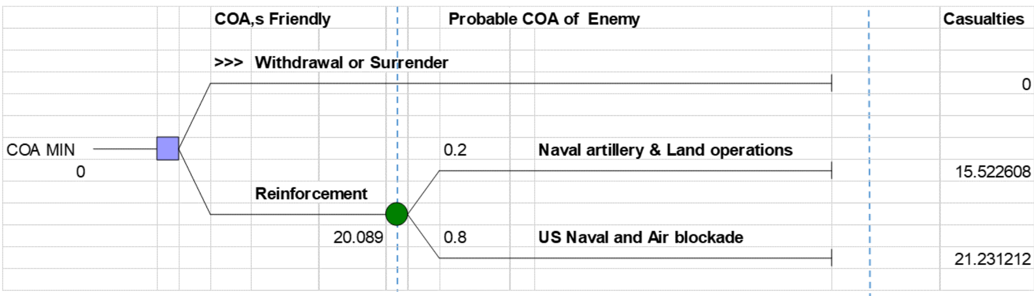

Figure 4.13: Development of the various courses of action, each Japan COA creates likely reaction alternatives for the US strategy, quantified through its doctrine. The final expected value for each COA defines the best choice.

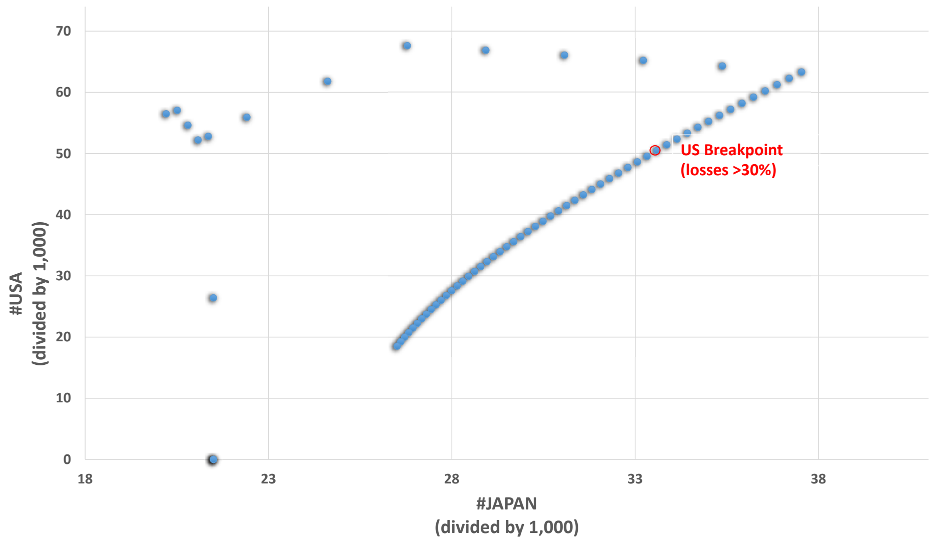

Figure 4.14: Scatter plot obtained from the predictive block for *Reinforcement and Naval artillery and land operations* as an alternative course, 20,000 Japanese reinforcement troops applied to prevent the invasion of the island following the 3:1 rule [Dav95]. The US breakpoint on the 14th day defines the Japan victory.

4.3 The Battle of Kursk

4.3.1 Historical overview

The Battle of Kursk, also called Operation Citadel, gives its name to a series of armed clashes that took place on July 5, 1943, in the II WW context. The troops of the German army would make the last offensive effort on the Soviet front, massing the main part of its armored forces to face the troops of the Red Army of the Soviet Union. See Figure 4.16.

Although a massive assault against Soviet troops with heavy tanks, ar-

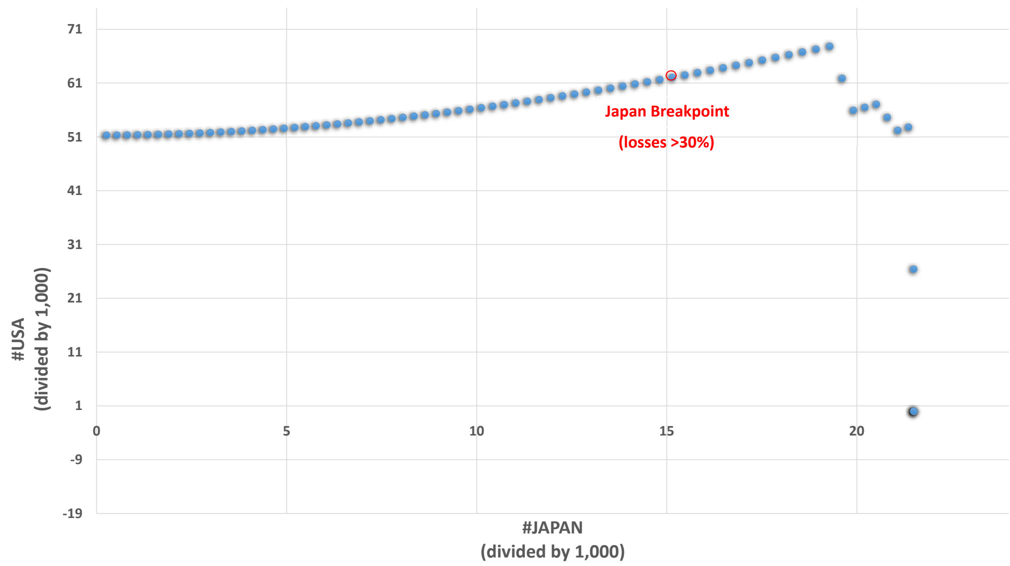

Figure 4.15: Scatter plot obtained from the predictive block for *Reinforcement and US Blockade* as an alternative COA. This shows how the COA does not prevent the invasion of the island due to the American strategy of blocking supply lines and reinforcements.

tillery, and aviation was planned, Adolf Hitler's postponements gave the Soviets time to prepare for defense.

According to the historical data:

- German planned *blitzkrieg* attacks north and south of the Kursk bulge.
- The battle period in this study is 14 days.
- German attack from July 5 to July 11.
- Soviet attack from July 12 to July 18.

- The Germans outnumbered the Soviets when they attacked, following the 3:1 law, [Dav95].
- Evolution of active combat troops and casualties according to [Bau98] report.

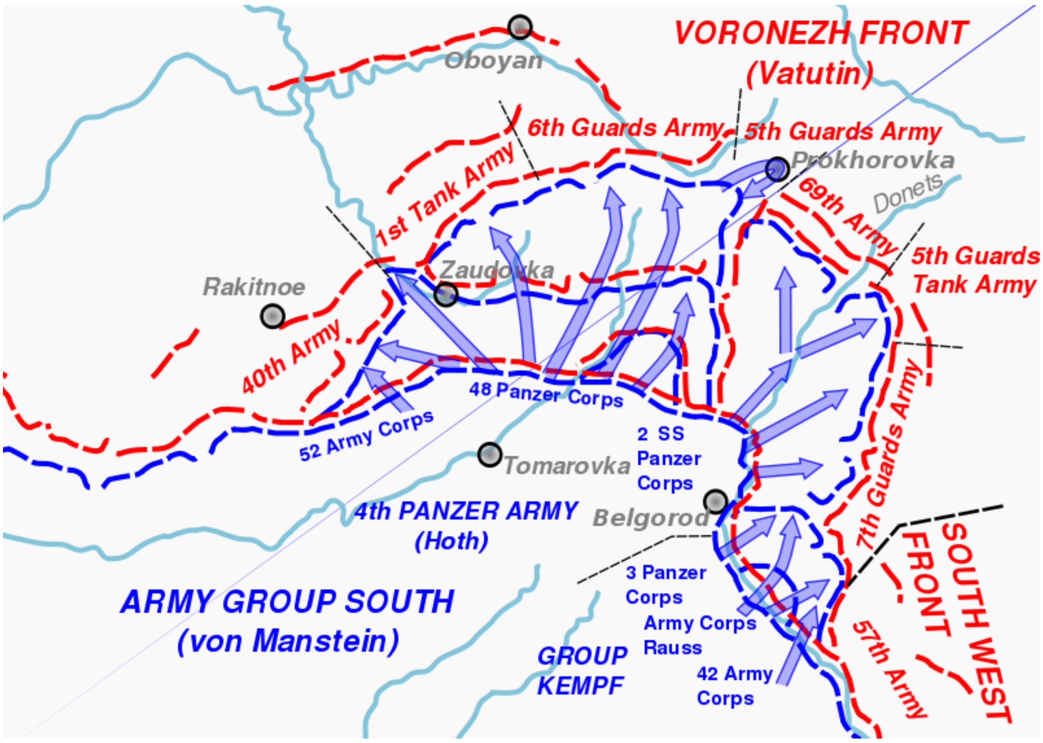

Figure 4.16: The German advance in the South was provided by New World Encyclopedia.

4.3.2 Battle Analysis

The Battle of Kursk has a very peculiar structure and complex development that is very interesting to discuss the performance provided by adaptive and predictive control and expert block in data fitting. All analyses that follow are based on the aggregated data about the southern side of the battle for combat units exposed to fire.

Dataset

Table 4.5 and Figure 4.17 describe the dataset regarding the southern front of the Kursk battle. This dataset was obtained from the Kursk Data Base (KDB) documented in the Kursk Operation Simulation and Validation Exercise-Phase II (KOSAVE II) report, [Bau98], and combined study of the following literature sources: [LD04], [Tur11] and [Spe11]. A negative value in the 'reinforcements' column means that a set of troops withdrew from combat.

Predictive Block

Given the values in Table 4.5, the predictive and adaptive blocks define which parameter values (p, q, a, and b) fit best the data evolution, according to the procedure depicted in Sections 3.2.2 and 3.5. The adaptive block sampling strategy changes concerning the previous case, Figure 4.18, seeking to feature battle phases on a big scale campaign, such as the battle of Kursk, by discarding the previous samples. Thus, the parameter values are adjusted to time windows homogeneously distributed within the dataset, see Figure 4.19.

The adjusted parameters clearly define two main phases, the German attack phase I, and the Soviet attack phase II.

- In phase I, the adaptive block defines p and q values close to the Lanchester logarithmic law. The German push was blocked by So-

	Total Number		Instant Change	
Days	S. troops	G. troops	S. reinforcement	G.reinforcement
1	84783	247866	0	0
2	141589	261368	56806	21770
3	163378	211212	25393	-41268
4	145875	227314	-14456	25000
5	179607	224664	36476	4884
6	166526	200686	-10458	-15370
7	219343	232938	54665	40390
8	252844	262920	35861	36616
9	175121	279697	-75148	34849
10	206465	208498	33375	-62511
11	89898	226075	-114890	23725
12	87769	131800	-1065	-91803
13	37981	149438	-49319	19752
14	119346	188079	81860	39098

Table 4.5: Battle of Kursk data set for units actually in combat (FCUD) gathered from the KOSAVE report available on https://apps.dtic.mil/sti/citations/ADA360311 (last visited July 20, 2021) and [LD04]. S and G stand for Soviet and German.

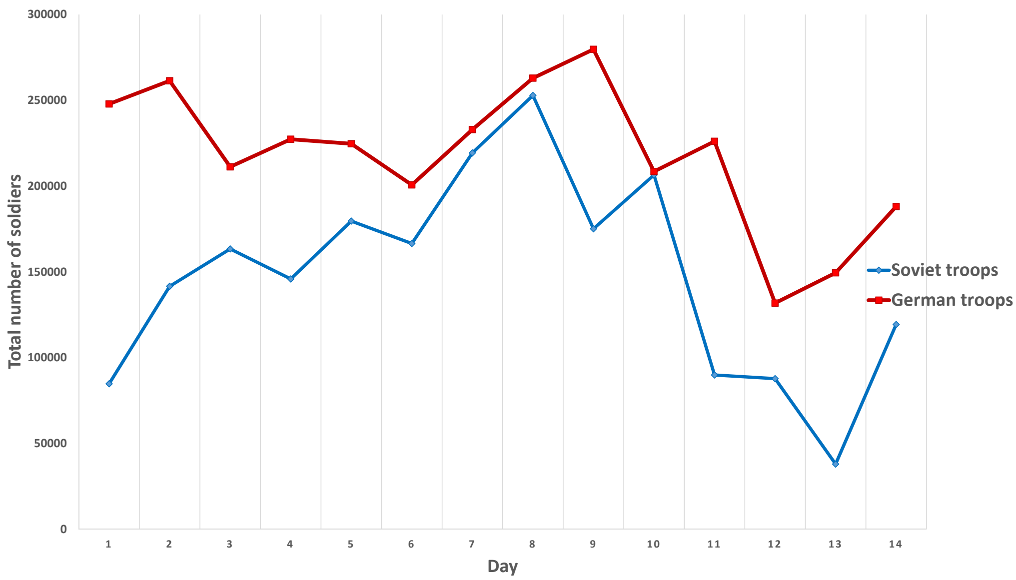

Figure 4.17: Evolution of Soviet and German troops from July 5, 1943, to July 18, 1943, on the southern side. The Battle of Kursk was the largest tank battle of WWII, took place in the area around the city of Kursk, and concluded with the defeat of the Germans.

viet mines and other anti-tank defenses. The casualties were proportional to own forces exposed to fire. The Soviet lethality coefficient b is high in this phase as they are in a *defensive position on fortified terrain*, compared to the average values obtained by [LD04].

- In phase II, the adaptive block defines p and q values close to the Lanchester square law. The casualties are proportional to enemy forces, each element in the battle with the ability to produce casualties knows the location of its targets and moves to a new target when the previous target is destroyed. At the end of the battle, the German

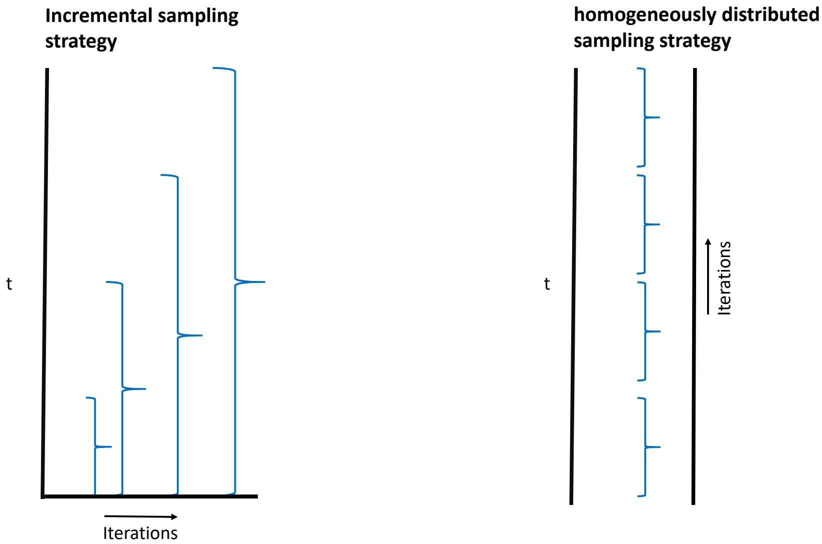

Figure 4.18: Sampling strategies for the adaptive block. The incremental sampling strategy determines the evolution of trend, by identifying variability associated with changes in combat typologies or *lethality*. In the case of a great battle (a big scale), a homogeneously distributed sampling strategy determines the evolutionary phases, although it will be more affected by outliers.

lethality coefficient a dropped due to the fatigue and motivation of the troops, agreeing on [RC16] general conclusion.

This sequence of phases coincides with the analysis of the actual casualties performed and historical events, see Figure 4.20.

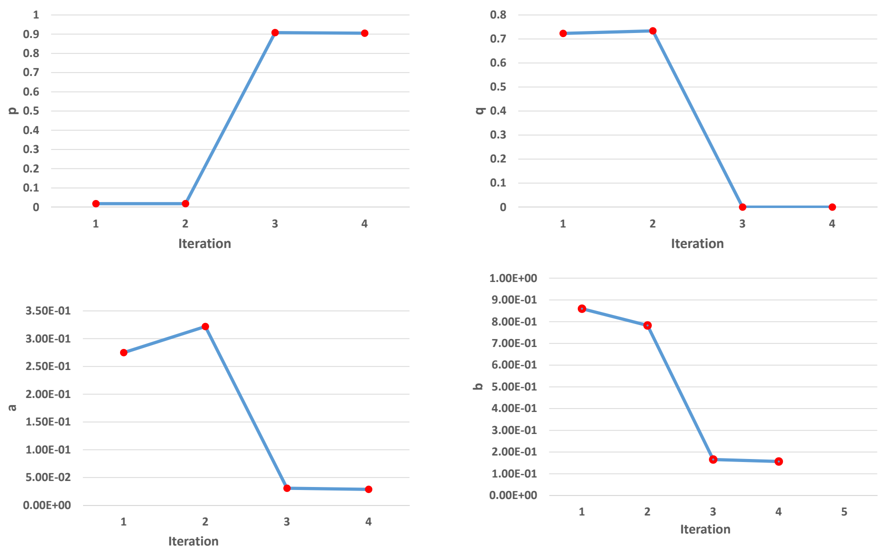

Figure 4.19: Parameter values evolution by adaptive block coincides with the historical battle phases, according to [Tur11].

Authors, such as [LD04] and [Tur11], found that no Lanchester model fitted the results based on the set of combat units exposed to fire in the battle of Kursk. Furthermore, [LD04] showed that by dividing the battle into phases properly, it is possible to attain a better fit of the Lanchester model to the data. This last assumption is supported by this analysis.

Expert Block

Following the procedure described in Section 3.3.1 identifies the probability evolution of success P that German troops need to launch an attack through the Soviet defender's Advantage Factor, $P_{Germantroops} < (1 - WinsDef)$. See Figure 4.21.

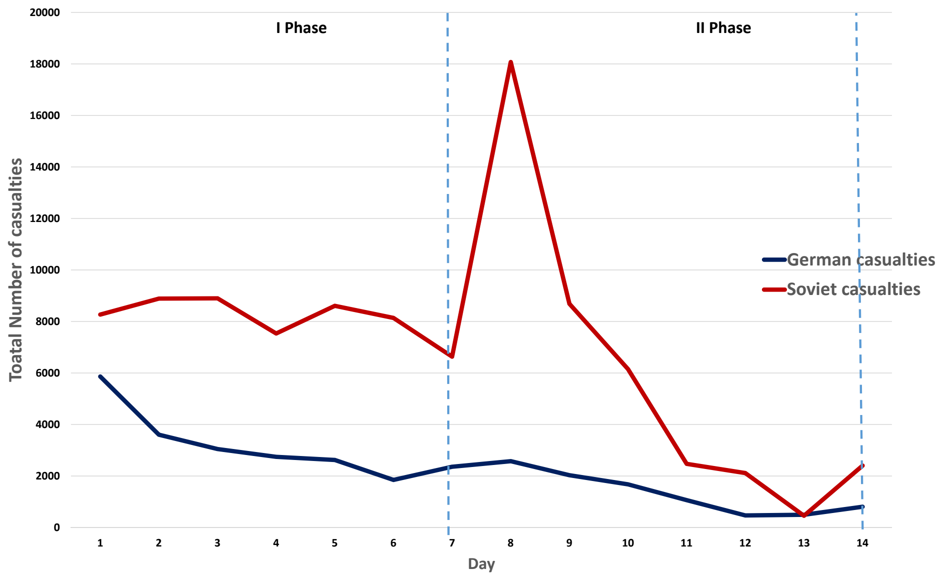

Figure 4.20: Evolution of the actual casualties of Soviet and German troops, which determines two main phases in the battle.

The German offensive failed, although the Soviet troops suffered four times more attrition than the Germans. The Germans changed their strategy to a *Defensive position* due to the series of Soviet counterattacks.

4.4 Concluding remarks

Casualty data of current battles are generally confidential and thus are not available to the general public. For this reason, the approach has been validated with well-documented World War II battles, in which the type of combat is primarily land-based where Lanchester models provide an effective framework, see Related work chapter 2. This type of combat has not

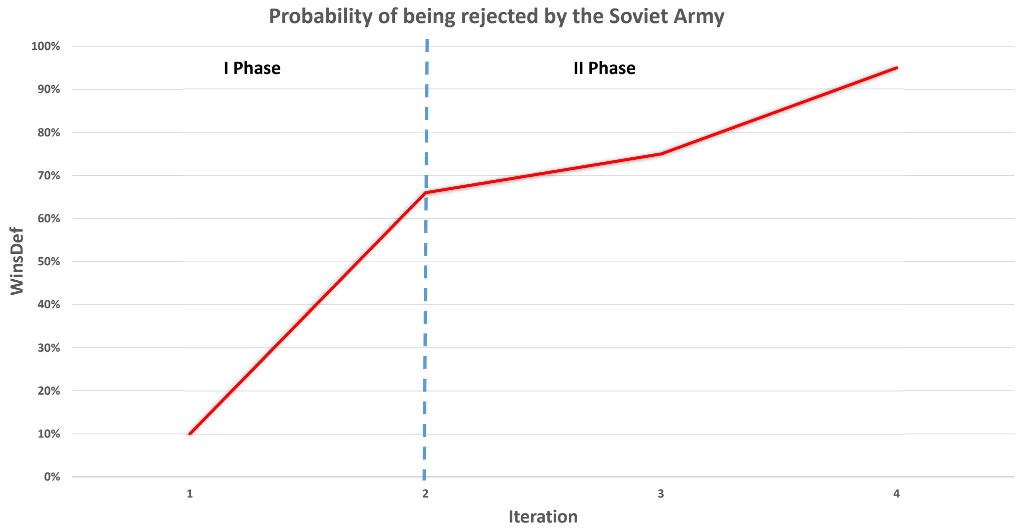

Figure 4.21: Analysis of the evolution of the probability of being rejected by the Soviet army by the expert block. It coincides with the real evolution of the battle, in the first phase the German troops executed an offensive strategy that finally is modified to defensive due to the Soviet counterattacks.

changed significantly since then due to technological or doctrinal changes for combat units exposed to fire following the philosophy of Lanchester's seminal work, [Lan16].

After the assessments of Crete, Iwo Jima, and Kursk World War II battles, it was possible to experimentally test the evolution of battle events if other decisions had been made in the theater of operations and even reproduce complex battle dynamics, based on the novel use of control theory in battle modeling to promote the capacity the ability to anticipate as an application in the automation of decisions high-level resolution.

5 Conclusions and future work

Due to the complexity of the military background, it is essential to make research advances and development efforts to enable decision automation, thus increasing the ability to anticipate the consequences of the adversary's possible actions and counteractions by new approaches and techniques that support the decision-making and the assessment under uncertainty. This presents a challenge for the upcoming battlefield, where the sophistication and scale of operations are expected to exceed the command's largely manual decision assessment capability. Section 5.1 summarizes this thesis main conclusions. Then, Section 5.2 gives directions for future work.

5.1 Conclusions

Lanchester's classic work on battle dynamics modeling has inspired important research on the development of combat abstractions to support military decision-making under conditions of uncertainty, pursuing ways to achieve combat superiority. Nevertheless, it has been subject to the following criticisms:

- It does not provide a fitting good enough for historical battle data.

- It uses a constant lethality factor.
- It deals with large battles with multiple types and phases as a whole.
- It performs an over-simplistic one-sided treatment without taking into account opponent's capabilities.
- It cannot be used for disaggregated engagements.

New models and techniques have emerged to expand the capabilities and to reduce the shortcomings of previous approaches, but they have failed to provide an adequate benchmark for high-level decision-making.

To face those criticisms and overcome the current state, this thesis has proposed a model that is focused on the types of decisions supported, how these types of decisions were made, and understanding the battle as a cause-effect process that evolves subject to changes and external actions. Thus, our framework removes the limitations of Lanchester's classic work by dynamically adjusting the factors that define the evolution of the land battlefield, including learning mechanisms that optimize the capabilities of the architecture and, in short, the ability to improve decisions under uncertainty.

In the thesis, we have provided empirical evidence showing that our framework fits land battle trends adequately and can select the most appropriate COA. As a result, our approach contributes to existing research by supporting decision-making at a high command level.

Besides, our framework promotes the military panel training by enabling the evolution of battle events to be tested if other decisions had been made and the automation of high-level decisions of the opposing force in Computer-assisted exercises (CAX), which is an indispensable part of the regular NATO training and operation exercises.

Finally, the strengths and weaknesses of existing decision-making models have been identified, and a model hierarchy has been drawn up to provide a reference background for the application of battle prediction models in decision-making.

5.2 Future work

Currently, our framework assumes that the cause-effect relationship of the battle can be modeled. The ways of including random variation were explored in Section 3.2.1, showing that there is no difference in qualitative terms. However, there may be a chaotic behavior in the final phases that makes such modeling difficult in some battles. Future work could try to apply artificial intelligence techniques to overcome this problem. Additionally, it could be considered incorporating into our model additional factors that may or not depend on the size of the forces. These factors could vary around fixed values as a function of the noise presented by confusion, momentum, and combat stress.

Moreover, the capabilities of some constituent blocks in the architecture could be expanded. For example, in the case of the *scheduler block*, we are working on determining enemy disposition patterns that allow us to estimate detailed tactical intentions, i.e., in a land combat scenario, the superior unit will tend to deploy its subordinate units on the terrain according to its doctrine and a pattern that allows it to accomplish an ordered mission. Once the pattern is recognized, it acts as an indicator of possible future movements and intentions. We have already proposed two technical approaches based on distance intra-units and typology, on the one hand using the Pearson correlation coefficient as a measure of variability when similarity is measured in terms of pattern or shape, and on the other hand, solving it using supervised neural networks.

Appendices

A
Lanchester's equations for combat

Lanchester's models are defined as Ordinary Differential Equations (ODEs) that support the prediction of confrontation results.

Table A.1 summarizes the most prominent Lanchester's equations for combat, assuming no reinforcements, according to their combat type and the degree of command and control maintained by the command of the situation. The following combat types are considered:

- *Direct (aimed) Fire*: Each member of the x-force is within the range of the enemy and, when the x-force receives casualties, the fire is concentrated on the remaining ones. See [Lan16].

- *Fire Concentrated in areas*: In the case of forces distributed in areas invisible to the enemy or using concentrated fires in areas such as artillery, the model of casualties of the x-force must be proportional to the size of the x-force. See [Lan16].

- *Combat Asymmetric*: battles between conventional x-force forces against guerrilla y-forces (invisible to the enemy). See [Dei62].

- *Unequal sized forces*: The difference in size between the contenders is a factor that conditions the lethality, inefficiencies of scale. Therefore, Helmbold [Hel65] added E_x and E_y functions that modify force *lethality* by a x and y ratio.

- *Great Battle*: A campaign on a big scale, i.e., an aggregation of many smaller battles. See [Fri98].

Interestingly, there is another interpretation of Lanchester's laws, already identified in Table A.1, related to the level of control and command of the forces and base on the difference in the degree of control maintained by the command of the situation; so where both forces have efficient command and control, the square law favors the side with superior numbers. Where both have inferior command and control, the linear law favors the force with better individual performance. Where both forces are grouped and become easy targets for the enemy, the logarithmic law favors the smaller force.

		Combat Type				
		Direct Fire *(Square Law)*	**Fire concentrated in areas** *(Linear Law)*	**Asymmetric combat**	**Unequal-Size Forces**	**Great Battles** *(Logarithmic Law)*
Control & Command Level	**Efficient**	$\frac{dx}{dt} = -ay(t)$ $\frac{dy}{dt} = -bx(t)$			$\frac{dx}{dt} = -ay(t)E_y\left(\frac{x}{y}\right)$ $\frac{dy}{dt} = -bx(t)E_x\left(\frac{y}{x}\right)$	
	Not so efficient		$\frac{dx}{dt} = -ay(t)x(t)$ $\frac{dy}{dt} = -bx(t)y(t)$	$\frac{dx}{dt} = -ay(t)$ $\frac{dy}{dt} = -bx(t)y(t)$		
	Poor					$\frac{dx}{dt} = -ax(t)$ $\frac{dy}{dt} = -by(t)$

Table A.1: Summary of Lanchester's equations for combat

B WinsDef Curve, base on analysis of historical battle

[Hel97] related the idea of the defender's advantage factor parameter (v) to the probability of rejecting the enemy's attack. This parameter is well-supported in 290 land battles in which about 83% of them are correctly classified through who won, taken from CDB90DAT available on https://github.com/jrnold/CDB90 (last visited June 26, 2021), ranging from 1600 to recent times.

Thus, [Hel97] identified the v parameter as a key parameter strongly related to success in battle.

B.1 WinsDef Curve

[Hel97] using the Lanchester's Square Law defined the v parameter as the relationship between the strength ratio of both contenders multiplied by their lethality factors, Equations B.2, B.2 and B.3, where $x(0)$ and $y(0)$ are the numbers of combatants of the x-force attacker and y-force defender at the initial instant, a is the *lethality* of the defender force, and equivalently b of the attacker.

$$v = \ln\sqrt{\frac{\delta}{\alpha}} \tag{B.1}$$

$$\alpha = b\left(\frac{x_0}{y_0}\right) \tag{B.2}$$

$$\delta = a\left(\frac{y_0}{x_0}\right) \tag{B.3}$$

Helmbold's experiment [Hel97] has been replicated in this thesis to trace the relationship between the defender's advantage factor parameter and the probability of rejecting the enemy's attack, using this time a new approximation by the logistic regression method and a simpler data set.

B.1.1 Dataset

From the CD90DAT data set, where a simpler subset was chosen, following the procedure of putting it in order by v and taking a representative sample using a non-parametric method such as systematic random sampling. See Table B.1.

B.1.2 Logistic regression

From the analysis of Table B.1, a relationship arises that through the simple logistic regression method allows estimating the probability of a binary variable as a function of a quantitative variable. See Figure B.1 and its performance measurement for classification problems by the graphical representation of sensitivity versus specificity for a binary classifier system, where sensitivity measures the rate of correct positive classification (1) and specificity the rate of correct negative classification (0), Figure B.2.

row	v	won the defender
1	-1.57	0
2	-1.29	0
3	-1.12	1
4	-0.88	0
5	-0.81	0
6	-0.72	0
7	-0.64	0
8	-0.59	0
9	-0.54	0
10	-0.51	0
11	-0.43	0
12	-0.37	0
13	-0.33	0
14	-0.28	0
15	-0.22	0
16	-0.13	1
17	-0.06	0
18	-0.01	1
19	0.01	0
20	0.05	0
21	0.07	1
22	0.17	0
23	0.24	1
24	0.33	1
25	0.41	1
26	0.48	1
27	0.65	1
28	0.9	1
29	1.16	1

Table B.1: Simple list of v against, a binary representation of who won from CD90DAT (1 if the defender won and 0 otherwise).

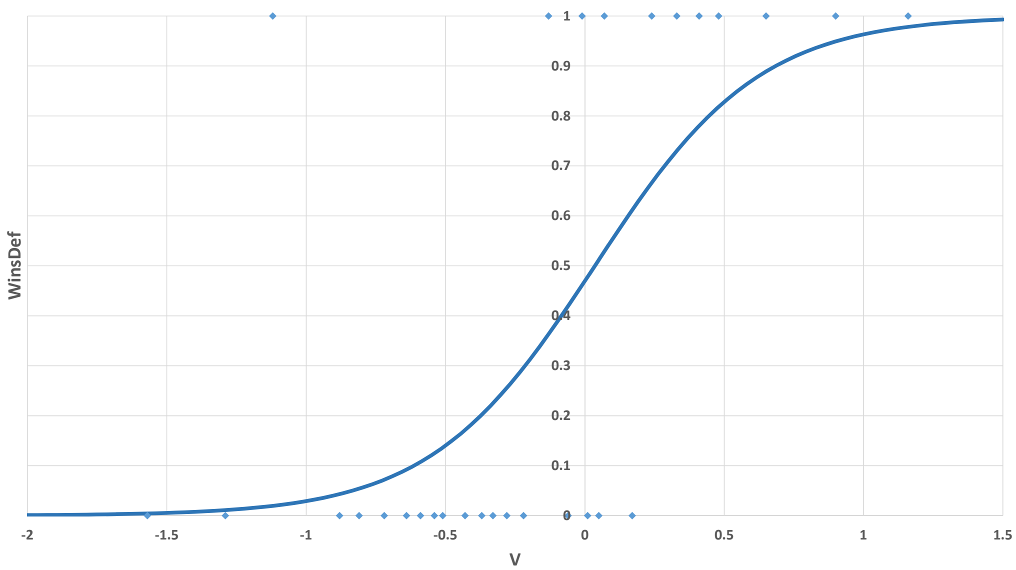

Figure B.1: WinsDef curve obtained by logistic regression, we can observe that v is positive when the defender has an advantage and negative when the attacker has an advantage.

This curve allows estimating the defender probability of success or failure, Equation (B.4).

$$\text{WinsDef} = \frac{1}{1 + e^{0.12 - 3.38v}} \tag{B.4}$$

Whenever attrition is ruled by square law, the v parameter can be a good index of defender superiority, in our thesis we have extended this conclusion to the rest of Lanchester's laws, obtaining results that coincide with real facts, for example, the case of the battle of Kursk. However, further research is needed.

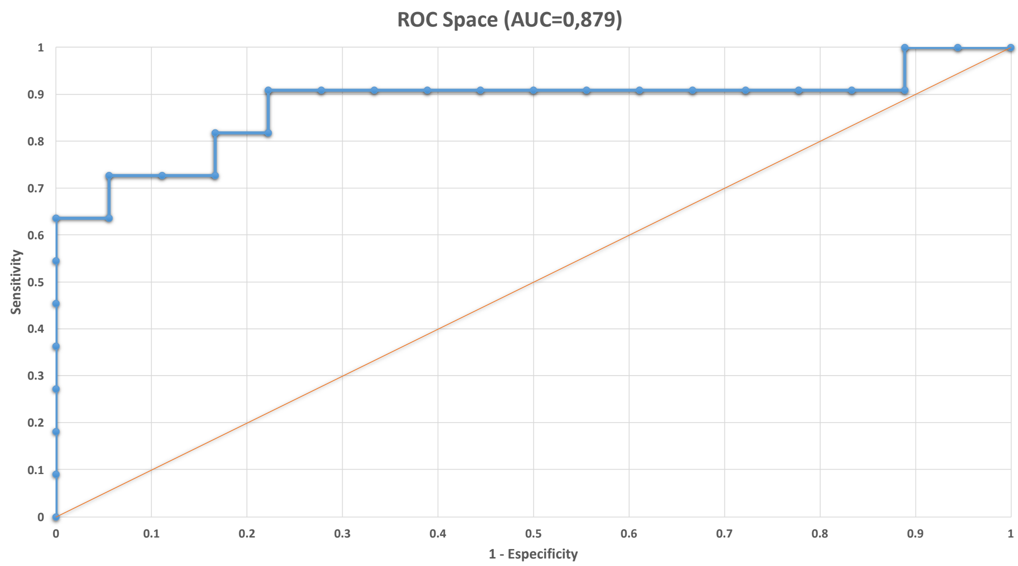

Figure B.2: This graph shows the performance of a classification model in which the area under the curve (AUC) factor is close to 1, indicating good accuracy.

C Calculation procedure

This appendix describes the main methods and calculation procedures used in solving Ordinary Differential Equations (ODEs).

C.1 Solving ordinary differential equations

Lanchester's models are defined as Ordinary Differential Equations (ODEs) that allow predicting the results of the confrontation. These ODEs are easily computable by numerical approximations, among the most frequent numerical approximations to solve ODEs are the so-called discretization methods that consist in finding the approximate solutions, see Figure C.1. In this thesis, the Euler method has been selected to approximate solutions to differential equations. Thus, by applying to the generalized model defined by [Bra95], Equations 3.19 and 3.19, where the values in the initial condition are also known numbers, we can obtain Equations C.1 and C.2.

$$x_{n+1} = x_n - a\frac{1}{d}({y_n}^p {x_n}^q)(t_{n+1} - t_n) \tag{C.1}$$

$$y_{n+1} = y_n - bd({x_n}^p {y_n}^q)(t_{n+1} - t_n) \tag{C.2}$$

Furthermore, we assume that the step sizes between the points are of uniform size equal to h ($t_{n+1} - t_n = h$).

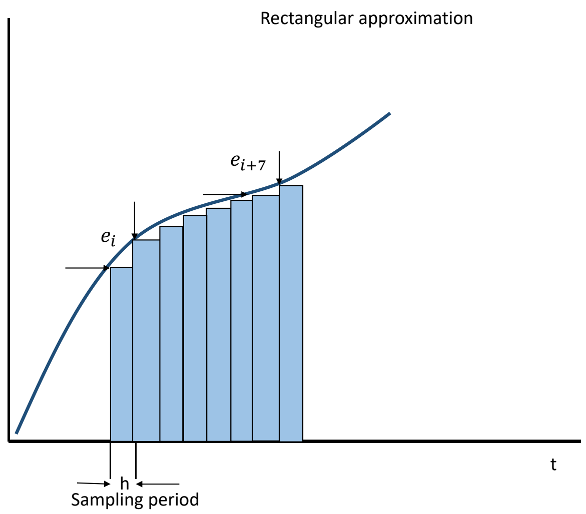

Figure C.1: Forward Euler rectangular approximation.

Following this discretization method, the attrition values are estimated for the studied time period.

On the other hand, in the case of the historical values previously defined, we can obtain the lethality factors for different periods under study by numerical integration, using Riemann sum, thus applying the Square law Equations 3.4 and 3.3, the following Equations, C.3 and C.4, are obtained.

$$y(t) - y_0 = b \int_{x=0}^{x=t} x(t)dt \simeq b \sum_{i=0}^{n} x_i h \tag{C.3}$$

$$x(t) - x_0 = a \int_{y=0}^{y=t} y(t)dt \simeq a \sum_{i=0}^{n} y_i h \tag{C.4}$$

where the step sizes between the points are of uniform size.

Other more accurate numerical calculation methods could be considered, but the range of accuracy obtained is sufficient for the case studies carried out in which the trend defines the decision.

D

GRG initial values and Set-point

This appendix describes the initial feasible values of the generalized model [Bra95] to start the GRG algorithm and the different values defining the set-point, which are the basis for control actions. The set-point determines the criterion, Table D.1, on which the control acts by generating a feasible COA to change the negative trend.

		Initial values					Set-point		
Battle	**Focos on**	**p**	**q**	**a**	**b**	**FCUD**	**Mission**	**Breakpoint**	**A. Limits**
Creta	Allied Side	1	0	0.0162	0.104	40,550	Reject the invasion	< 30%	NA
Iwo-Jima	Japan Side	1	0	0.0554	0.0106	21,500	Reject the invasion	< 30%	NA
Kursk	Soviet Side	1	1	0.0004048	0.0001236	84,783	Control (Eastern Front)	NA	NA

Table D.1: The generalized model initial values and target set-point. The set-point follows guidelines; *Accomplish the assigned mission at a reasonable cost*. $FCUD$ stands for the Fighting Combat Unit Data, NA stands for Not Applicable, and A stand for Attrition.

References

[Aba78] J. Abadie. The grg method for nonlinear programming. *Design and implementation of optimization software*, pages 335–362, 1978.

[ADK17] S. Ajitha, A. Datta, and T. V. S. Kumar. Multi-Agent based Artificial War. In *International Conference on Advanced Computing (ICoAC)*, Chennai, India, 2017.

[AGK11] M. P. Atkinson, A. Gutfraind, and M. Kress. Lanchester theory and the fate of armed revolts. *arXiv preprint arXiv:1106.4358*, 2011.

[Ale94] J. H. Alexander. *Closing in: Marines in the seizure of Iwo Jima*. History and Museums Division, Headquarters, US Marine Corps, 1994.

[And93] L. B. Anderson. A heterogeneous shoot-look-shoot attrition process. *Simulation and Gaming*, 24(3):277–293, 1993.

[Arm14] M. J. Armstrong. Modeling short-range ballistic missile defense and israel's iron dome system. *Operations Research*, 62(5):1028–1039, 2014.

[AS15] M. J. Armstrong and S.E. Sodergren. Refighting pickett's charge: Mathematical modeling of the civil war battlefield. *Social Science Quarterly*, 96(4):1153–1168, Dec 2015.

[Bar69] C. B. Barfoot. The lanchester attrition-rate coefficient: Some comments on seth bonder's paper and a suggested alternate method. *Operations Research*, 17(5):888–894, 1969.

[Bau98] W. J. Bauman. Kursk operation simulation and validation exercise - phase ii (kosave ii). Technical report, Army Concepts Analysis Agency Bethesda MD, Sep 1998.

[Bia14] M. A. Biank. *Battle Of Crete: Hitler's Airborne Gamble*. Pickle Partners Publishing, Auckland, New Zealand, Aug 15, 2014.

[BK14] P. Bholowalia and A. Kumar. Ebk-means: A clustering technique based on elbow method and k-means in wsn. *International Journal of Computer Applications*, 105(9), 2014.

[BKR95] J. Bracken, M. Kress, and R.E. Rosenthal. *Warfare modeling*. Military Operations Research Society, 1995.

[Bon67] S. Bonder. The lanchester attrition-rate coefficient. *Operations Research*, 15(2):221–232, 1967.

[Bra95] J. Bracken. Lanchester models of the Ardennes campaign. *Naval Research Logistics*, 42(4):559–577, 1995.

[BS13] O. Batarseh and D. Singham. Interval-based simulation to model input uncertainty in stochastic Lanchester models. *Military Operations Research*, 18(1):61–75, 2013.

[Cha16] L.N.P. Chan. Lanchester square law: Its implications for force structure and force preparation of singaporeâĂŹs operationally- ready soldiers. *J. Singap. Armed Forces 42*, pages 47–60, 2016.

[Che03] H. M. Chen. An optimal control problem in determining the optimal reinforcement schedules for the Lanchester equations. *Computers and Operations Research*, 30(7):1051–1066, 2003.

[Che07] H. Chen. A non-linear inverse Lanchester square law problem in estimating the force-dependent attrition coefficients. *European Journal of Operational Research*, 182(2):911–922, 2007.

[Chr95] T. Christensen. Stable defence; technical report ac/243 (panel 7) tr/5. *NATO Defence Research Group*, 1995.

[CJLL12] X. Chen, Y. Jing, C. Li, and M. Li. Warfare Command Stratagem Analysis for Winning Based on Lanchester Attrition Models. *Journal of Systems Science and Systems Engineering*, 21(1):94–105, 2012.

[Cou19] S. G. Coulson. Lanchester modelling of intelligence in combat. *IMA Journal of Management Mathematics*, 30(2):149–164, 2019.

[Cox01] K. J. Cox. The battle for crete (operation mercury): An operational analysis. Technical report, Naval War Coll Newport RI Joint Military Operations Dept, May 18, 2001.

[CQ14] X. Chen and J. Qiu. Differential game for a class of warfare dynamic systems with reinforcement based on Lanchester equation. *Abstract and applied analysis*, 2014:1–8, Apr 22, 2014.

[Dav95] P. K. Davis. An introduction to variable-resolution modeling and cross-resolution model connection (R-4252-DARPA). Technical report, RAND Corporation, 1995.

[DB91] P. K. Davis and D. Blumenthal. The base of sand problem: A white paper on the state of military combat modeling. Technical report, RAND Corporation, 1991.

[DDF+16] H. Dawid, K. F. Doerner, G. Feichtinger, P. M. Kort, and A. Seidl. *Dynamic Perspectives on Managerial Decision Making*, volume 22. Springer, Cham, 1st ed. 2016 edition, 2016.

[Dei62] S. J. Deitchman. A Lanchester Model of Guerrilla Warfare. *Operations research*, 10(6):818–827, 1962.

[Din01] J. A. Dinges. *Exploring the validation of Lanchester equations for the Battle of Kursk*. PhD thesis, Naval Postgraduate School, Monterey, California, 2001.

[Duf17] R. B. Duffey. Dynamic theory of losses in wars and conflicts. *European Journal of Operational Research*, 261(3):1013–1027, 2017.

[Eng54] J. H. Engel. A Verification of Lanchester's Law. *Journal of the Operations Research Society of America*, 2(2):163–171, 1954.

[Eng63] J. H. Engel. Combat effectiveness of allied and german troops in the world war 2 invasion of crete. Technical report, Center for Naval Analyses Alexandria VA, Operations Evaluation Group, Feb 8, 1963.

[Eps85] J. M. Epstein. *The Calculus of Conventional War: Dynamic Analysis Without Lanchester Theory*. Brookings Institution Press, 1985.

[Eps88] J. M. Epstein. Dynamic Analysis and the Conventional Balance in Europe. *International Security*, 12(4):154–165, 1988.

[Fri98] R. D. Fricker. Attrition models of the Ardennes campaign. *Naval Research Logistics*, 45(1):1–22, 1998.

[GJ97] D. P. Gaver and P. A. Jacobs. Attrition modeling in the presence of decoys: An operations-other-than-war motivation. *Naval Research Logistics*, 44(5):507–514, 1997.

[GJCJMO14] V. Gomez-Jauregui, C.Gomez-Jauregui, C. Manchado, and C. Otero. Information management and improvement of citation indices. *International Journal of Information Management*, 34(2):257 – 271, 2014.

[Gla03] D. Glantz. *The Soviet Strategic Offensive in Manchuria, 1945*, volume 7. Routledge, London, 2003.

[GV11] E. Gonzalez and M. Villena. Spatial Lanchester models. *European Journal of Operational Research*, 210(3):706–715, 2011.

[HD87] T. F. Homer-Dixon. A common misapplication of the Lanchester square law: A research note. *International Security*, 12:135 – 139, 1987.

[Hel61] R. L. Helmbold. Historical data and Lanchester's theory of combat (CORG-SP-128). Technical report, Combat Operations Research Group, Technical Operations, Inc., 1961.

[Hel65] R. L. Helmbold. A modification of Lanchester's equations. *Operations research*, 13(5):857–859, 1965.

[Hel97] R. L. Helmbold. The advantage parameter: A compilation of phalanx articles dealing with the motivation and empirical data supporting use of the advantage parameter as a general measure of combat power (caa-mr-97-3). Technical report, The US Army's Center for Strategy and Force Evaluation, 1997.

[HH95] D. S. Hartley and R. L. Helmbold. Validating Lanchester's square law and other attrition models. *Naval Research Logistics*, 42(4):609–633, 1995.

[HM00] K. Hausken and John F. Moxnes. The microfoundations of the lanchester war equations. *Military Operations Research*, 5(3):79–99, 2000. 17.

[HML17] W. Hu, D. Mei, and Y. Liu. Generalized Lanchester warfare model characterized by information asymmetry effect. *Journal of Systems Engineering and Electronics*, 28(2):315–321, 2017.

[HPB+91] D.S. HartleyIII, K.J. Posen, B.R. Bader, J. Morstein, and J.R. Brinkerhoff. Forecasting relative combat effectiveness: A feasibility study. Technical report, MARTIN MARIETTA ENERGY SYSTEMS INC OAK RIDGE TN, 1991.

[IFW91] L. Ingber, H. Fujio, and M. F. Wehner. Mathematical comparison of combat computer models to exercise data. *Mathematical and Computer Modelling*, 15(1):65–90, 1991.

[IHJ19] N.S. Intizhami, A.Y. Husodo, and W. Jatmiko. Warfare simulation:predicting battleship winner using random forest. In *IEEE International Conference on Communication, Networks and Satellite (Comnetsat)*, 2019.

[Ila00] A. Ilachinski. Irreducible semi-autonomous adaptive combat (ISAAC): An artificial-life approach to land combat. *Military Operations Research*, 5(3):29–46, 2000.

[Ila01] A. Ilachinski. Cellular automata : a discrete universe. *Singapore ; World Scientific, c2001*, 2001.

[JHC17a] F. Jisong, R. Hui, and T. Chengxiang. An analysis of wargame rules simulation based on stochastic lanchester models. In *International Conference on Network, Communication and Computing (CNCC 2017)*, ICNCC 2017, pages 135–139. ACM, Dec 8, 2017.

[JHC17b] Fan Jisong, Ren Hui, and Tian Chengxiang. An Analysis of Wargame Rules Simulation Based on Stochastic Lanchester Mod. In *International Conference on Network, Communication and Computing (ICNCC)*, Kunming, China, 2017.

[JM11] I. R. Johnson and N. J. MacKay. Lanchester models and the battle of Britain. *Naval Research Logistics*, 58(3):210–222, 2011.

[JM15] D. D. P. Johnson and N. J. MacKay. Fight the power: Lanchester's laws of combat in human evolution. *Evolution and Human Behavior*, 36(2):152–163, 2015.

[Joh96] M. J. Johnson. Quantifying the value of reconnaissance using Lanchesterian type equations. *Mathematical and Computer Modelling*, 23(1-2):73–80, 1996.

[JYL+18] N. Jia, Z. Yang, T. Liao, Y. Dou, and K. Yang. A system dynamics model for analyzing swarming uavs air combat system. In *13th Annual Conference on System of Systems Engineering (SoSE 2018)*, pages 74–81, 2018.

[JZD17] Y. Jing, S. Zhang, and G. M. Dimirovski. Complexity of warfare command, communication and control systems simplified: Optimal resource partitioning via Lanchester equations. In *IEEE 17th International Conference on Smart Technologies (EUROCON 2017)*, pages 611–617, 2017.

[KBG+05] A. Kott, R. Budd, L. Ground, L. Rebbapragada, and J. Langston. Building a tool for battle planning: Challenges, tradeoffs, and experimental findings. *Applied Intelligence*, 23(3):165–189, 2005.

[KC07] B. Kitchenham and S. Charters. Guidelines for performing systematic literature reviews in software engineering. Technical report, Keele University and University of Durham, UK, 2007.

[KCF+18] M. Kress, J. P. Caulkins, G. Feichtinger, D. Grass, and A. Seidl. Lanchester model for three-way combat. *European Journal of Operational Research*, 264(1):46–54, 2018.

[Kir85] D. L. I. Kirkpatrick. Do Lanchester's equations adequately model real battles? *The RUSI Journal*, 130(2):25–27, 1985.

[Kle80] K. D. Klemm. *A further comparison of deterministic and stochastic Lanchester-type combat models*. PhD thesis, Naval Postgraduate School, Monterey, California, 1980.

[KLM18] M. Kress, K. Y. Lin, and N. J. MacKay. The attrition dynamics of multilateral war. *Operations research*, 66(4):950–956, 2018.

[KM14] M. Kress and Niall J. MacKay. Bits or shots in combat? the generalized deitchman model of guerrilla warfare. *Operations Research Letters*, 42(1):102–108, Jan 2014.

[KMPS17] D. Kim, H. Moon, D. Park, and H. Shin. An efficient approximate solution for stochastic lanchester models. *The Journal of the Operational Research Society*, 68(11):1470–1481, Nov 1, 2017.

[Kre20] M. Kress. Lanchester models for irregular warfare. *Mathematics (Basel)*, 8(5):737, May 7, 2020.

[KT99] M. Kress and I. Talmor. A new look at the 3 : 1 rule of combat through markov stochastic lanchester models. *Journal of the Operational Research Society*, 50(7):733–744, 1999.

[Lan16] F. W. Lanchester. *Aircraft in Warfare: the Dawn of the Fourth Arm*. Lanchester Press Inc., 1916.

[LD04] T. W. Lucas and J. A. Dinges. The effect of battle circumstances on fitting Lanchester equations to the battle of kursk. *Military Operations Research*, 9(2):17–30, 2004.

[Lep87] J. W. R. Lepingwell. The laws of combat? Lanchester reexamined. *International Security*, 12(1):89–134, 1987.

[LM14a] K. Y. Lin and N. J. MacKay. The optimal policy for the one-against-many heterogeneous lanchester model. *Operations research letters*, 42(6/7):473–477, 2014.

[LM14b] Kyle Y. Lin and N.J. MacKay. The optimal policy for the one-against-many heterogeneous Lanchester model. *Operations Research Letters*, 42(6-7):473–477, 2014.

[LS02] M. K. Lauren and R. T. Stephen. Map-aware Non-uniform Automata (MANA) a New Zealand Approach to Scenario Modelling. *Journal of Battlefield Technology*, 5(1):1, 2002.

[LT04] T. W. Lucas and T. Turkes. Fitting Lanchester equations to the battles of Kursk and Ardennes. *Naval Research Logistics*, 51(1):95–116, 2004.

[Luc00] Thomas W. Lucas. The stochastic versus deterministic argument for combat simulations: Tales of when the average won't do. *Military Operations Research*, 5(3):9–28, 2000. ID: Lucas00; M1: Journal Article.

[LWJR78] L. Lasdon, A. Waren, A. Jain, and M. Ratner. Design and testing of a generalized reduced gradient code for nonlinear programming. *ACM transactions on mathematical software*, 4(1):34–50, Mar 1, 1978.

[Mac06] Niall MacKay. Lanchester combat models. *Mathematics Today*, 42, 2006.

[Mac09] N. J. MacKay. Lanchester models for mixed forces with semi-dynamical target allocation. *Journal of the Operational Research Society*, 60(10):1421–1427, 2009.

[Mac15] N. J. MacKay. When Lanchester met Richardson, the outcome was stalemate: a parable for mathematical models of insurgency. *Journal of the Operational Research Society*, 66(2):191–201, 2015.

[MC19a] G. Minguela and C. Cerrada. Military operational decision maker based on blocks feedback, using uncertainty techniques. *Revista Iberoamericana de AutomÃątica e InformÃątica industrial (2019)*, 2019.

[MC19b] G. Minguela and C. Cerrada. Study of the combat model of lanchester as support for the construction of a military operational strategic decisor through feedback blocks. *Jornadas de AutomÃątica (JA-2019*, 2019.

[McN99] K. R. McNaught. The effects of splitting exponential stochastic Lanchester battles. *Journal of the Operational Research Society*, 50(3):244–254, 1999.

[Mil89] J. M. Miller. Crete and the three levels of war. Technical report, ArmyWar Coll Carlisle Barracks Pa, Feb 28, 1989.

[MK46] P.M. Morse and G. E. Kimball. Methods of operations research. Technical report, Center for Naval Analyses Alexandria VA Operations Evaluation Group, 1946.

[ML08] G. C. McIntosh and M. K. Lauren. Incorporating fractal concepts into equations of attrition for military conflicts. *Journal of the Operational Research Society*, 59(5):703–713, 2008.

[Mor46] C. P. Morehouse. *The Iwo Jima Operations*. Historical Division, Headquarters, US Marine Corps, 1946.

[Mor48] P. M. Morse. Mathematical problems in operations research. *Bulletin of the American Mathematical Society*, 54(7):602–621, 1948.

[MPW16] N. J. MacKay, C. Price, and A. J. Wood. Weight of shell must tell : A lanchestrian reappraisal of the battle of jutland. *History*, 101(347):536–563, Sep 25, 2016.

[MTL07] J. Y. Menq, P. C. Tuan, and T. S. Liu. Conceptual Lanchester-type decapitation warfare modelling. *Defence Science Journal*, 57(4):595–607, 2007.

[NAT11] NATO. Allied joint doctrine for the conduct of operationsÂů ajp-3(b), 2011.

[Off80] U.S. General Accounting Office. Models, data, and war: a critique of the foundation for defense analyses. Technical report, NATO, 1980. PAD-80-21, Washington, D.C.

[OT17] Da. Ormrod and B. Turnbull. Attrition rates and maneuver in agent-based simulation models. *Journal of Defense Modeling and Simulation-Applications Methodology Technology-Jdms*, 14(3):257–272, 2017.

[Pet67] R. H. Peterson. Letter to the editor – On the “logarithmic law” of attrition and its application to tank combat. *Operations Research*, 15:557–558, 1967.

[PSD89] V. Protopopescu, R. T. Santoro, and J. Dockery. Combat modeling with partial differential equations, 1989.

[R72] W. Samz R. Some comments on engel’s "a verification of lanchester’s law". *Operations Research*, 20(1):49–52, 1972.

[RC16] X. Rubio-Campillo. Model selection in historical research using approximate Bayesian computation. *Plos One*, 11(1):1–18, 2016.

[Rod89] F. C. Rodrigues. A proposed entropy measure for assessing combat degradation. *Journal of the Operational Research Society*, 40(8):789–793, 1989.

[SALA98] I. Sanchez-Alvarez and S. Lopez-Ares. OptimizaciÃşn con solver. *Rect@. Revista electrÃşnica de comunicaciones y trabajos de ASEPUMA*, Actas 6(1):29, Jan 1, 1998.

[SBB15] M. Stanescu, N. Barriga, and M. Buro. Using Lanchester attrition laws for combat prediction in starcraft. In *Conference on Artificial Intelligence and Interactive Digital Entertainment (AIIDE)*, 2015.

[Sch67] M. Schaffer. Lanchester models of guerrilla engagements. *RAND Corporation*, 1967.

[Sch07] M. B. Schaffer. A model of 21st century counterinsurgency warfare. *Journal of Defense Modeling and Simulation-Applications Methodology Technology-Jdms*, 4(3):252–261, 2007.

[Sch12] H. C. Schramm. Lanchester models with discontinuities: An application to networked forces. *Military Operations Research*, 17(4):59–68, 2012.

[SD15] M. Sahni and S. K. Das. Performance of maximum likelihood estimator for fitting lanchester equations on kursk battle data. *Journal of battlefield technology*, 18(2):23–30, Jul 2015.

[SG05] P. S. Sheeba and D. Ghose. Optimal resource partitioning in a military conflict based on Lanchester attrition models. *IEEE Conference on Decision and Control and European Control Conference (CDC 2005), Vols 1-8*, pages 5859–5864, 2005.

[SG06] P. S. Sheeba and D. Ghose. Optimal resource partitioning in conflicts based on lanchester (n, 1) attrition model. *American Control Conference (ACC 2006), Vols 1-12*, 1-12:6 pp.–, 2006.

[SG08a] P. S. Sheeba and D. Ghose. Optimal resource allocation and redistribution strategy in military conflicts with lanchester square law attrition. *Naval Research Logistics*, 55(6):581–591, 2008.

[SG08b] P. S. Sheeba and D. Ghose. Optimal resource allocation in conflicts with the lanchester linear law (2,1) model. *American Control Conference (ACC2008), Vols 1-12*, pages 1806–1811, 2008.

[SG13] H. C. Schramm and D.P. Gaver. Lanchester for cyber: The mixed epidemic-combat model. *Naval Research Logistics (NRL)*, 60(7):599–605, 2013.

[SJ16] D. Shlapak and M. W. Johnson. *Reinforcing Deterrence on NATOâĂŹs Eastern Flank*. RAND Corporation, Jan 22, 2016.

[SL18] C. R. Sugimoto and V. Lariviere. *Measuring Research: What Everyone Needs to Know*. Oxford University Press, 2018.

[SN14] H. C. Schramm and B.D. Nedialko. Differential equation models for sharp threshold dynamics. *Mathematical Biosciences*, 247:27–37, 2014.

[SOR+09] D. Shlapak, D. Orletsky, T. Reid, M. Tanner, and B. Wilson. *A Question of Balance: Political Context and Military Aspects of the China-Taiwan Dispute*. RAND, 2009.

[Spe11] L. R. Speight. Within-campaign analysis: A statistical evaluation of the battle of kursk. *Military operations research (Alexandria, Va.)*, 16(2):41–62, Jan 1, 2011.

[SR95] J.M. Sanchez and J. Rodellar. *Adaptive Predictive Control: From the concepts to plant optimization*. Prentice-Hall, London [u.a.], 1. publ. edition, 1995.

[Str11] J.S. Strickland. Mathematical modeling of warfare and combat phenomenon. *Inc., Colorado Springs CO*, 2011.

[Sym17] P. Syms. Validating Lanchester models: the first 60 years. In *International Symposium on Military OR (ISMOR)*, London, UK, 2017.

[Tam98] J. H. Tam. Application of Lanchester combat model in the Ardennes campaign. *Natural Resource Modeling*, 11(2):95–116, 1998.

[Tay79] J. G. Taylor. Attrition modelling (14961.7m). Technical report, Naval postgraduate School, 1979.

[Tay82] J. G. Taylor. Annihilation prediction for Lanchester-type models of modern warfare with logistics constraints. *Mathematical Modelling*, 3(4):323–340, 1982.

[TB78] J. G. Taylor and G. G. Brown. Numerical determination of parity-condition parameter for Lanchester-type equations of modern warfare. *Computers and Operations Research*, 5(4):227–242, 1978.

[TBBM01] G. Tecuci, M. Boicu, M. Bowman, and D. Marcu. An innovative application from the darpa knowledge bases programs - rapid development of a course of action critiquer. *Ai Magazine*, 22(2):43–61, 2001.

[Tol12] A. Tolk. *Engineering Principles of Combat Modeling and Distributed Simulation*. John Wiley and Sons, 2012.

[Tol16] A. Tolk. Tutorial on the engineering principles of combat modeling and distributed simulation. In *Winter Simulation Conference (WSC 2016)*, pages 255–269, Dec 2016.

[Tur11] T. Turkes. *Fitting Lanchester and other equations to the Battle of Kursk data*. PhD thesis, 2011.

[TV03] G. Trautteur and R. Virgilio. An agent-based computational model for the battle of trafalgar: a comparison between analytical and simulative methods of research. In *Twelfth IEEE International Workshops on Enabling Technologies: Infrastructure for Collaborative Enterprises (ET ICE 2003)*, pages 377–382. IEEE, 2003.

[TYM00] J. G. Taylor, U. Z. Yildirim, and W. S. Murphy. Hierarchy-of-models approach for aggregated-force attrition. In *Winter Simulation Conference WSC Proceedings (Cat. No.00CH37165)*, volume 1, pages 925–932 vol.1, 2000.

[VG09] E. S. Vieira and J. A. N. F. Gomes. A comparison of Scopus and Web of Science for a typical university. *Scientometrics*, 81(2):587–600, 2009.

[Wan14] M. Waniek. An agent-based simulation of the battle of kokenhausen. In *IEEE/WIC/ACM International Joint Conferences on Web Intelligence (WI) and Intelligent Agent Technologies (IAT),(WI-IAT 14)*, volume 3 of *WI-IAT '14*, pages 412–419. IEEE Computer Society, Aug 11, 2014.

[Was00] A. R. Washburn. Lanchester systems. Technical report, available on http://hdl.handle.net/10945/38390, Naval Postgraduate School, 2000.

[WFW95] S. Wrigge, A. Fransen, and L. Wigg. The lanchester theory of combat and some related subjects. a bibliography 1900-1993. Technical report, National Defence Research Establishment Stockholm (Sweden) Dept of Defence, 1995.

[WK09] Alan Washburn and Moshe Kress. *Attrion models*, pages 79–110. Springer US, Boston, MA, 2009.

[Woh14] C. Wohlin. Guidelines for snowballing in systematic literature studies and a replication in software engineering. In *International Conference on Evaluation and Assessment in Software Engineering (EASE)*, New York, NY, USA, 2014.

[WPY00] M. P. Wiper, L. I. Pettit, and K.D.S. Young. Bayesian inference for a lanchester type combat model. *Naval Research Logistics*, 47(7):541–558, 2000.

[WRdMSN+13] C. Wohlin, P. Runeson, P. Anselmo da Mota Silveira Neto, E. Engström, I. do Carmo Machado, and Eduardo Santana de Almeida. On the reliability of mapping studies in software engineering. *Journal of Systems and Software*, 86(10):2594 – 2610, 2013.

[WRH+12] C. Wohlin, P. Runeson, M. Höst, M. C. Ohlsson, B. Regnell, and A. Wesslen. *Experimentation in Software Engineering*. Springer-Verlag Berlin Heidelberg, 2012.

[WSWL12] Q. Wang, S. Shen, F. Wang, and Y. Liang. Research on battle agent model in the combat modeling. In *IEEE Symposium on Electrical and Electronics Engineering (EEESYM)*, pages 86–89. IEEE, Jun 2012.

[XZ14] X.Chen and A. Zhang. Modeling and optimal control of a class of warfare hybrid dynamic systems based on lanchester (n, 1) attrition model. *Mathematical Problems in Engineering*, 2014:1–7, 2014.

[YHJ13] G. Yang, K. Hung, and P. Julian. Adopting Lanchester model to the ardennes campaign with deadlock situation in the shift time between defense and attack. *International Journal of Information and Management Sciences*, pages Volume 24, Number 4, pp . 349–362, 2013, 2013.

[You72] L. W. Young. *Comparison of a Deterministic and a Stochastic Model for the Probability of Winning in a Two-Sided Combat Situation*. PhD thesis, Naval Postgraduate School, Monterey, California, 1972.

[Zha15] G. Zhang. Research of force requirement model with the organizational system of the army based on warfighting capability index. In *8th International Symposium on Computational Intelligence and Design (ISCID)*, volume 1, pages 9–12. IEEE, Dec 2015.