RESUMEN DE
TESIS DOCTORAL

2022

FORECASTING REALIZED DENSITIES: A COMPARISON OF HISTORICAL, RISK-NEUTRAL, RISK-ADJUSTED AND SENTIMENT-BASED TRANSFORMATIONS

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TOMAS PRIETO RUMEAU

A mis padres, por ayudarme a elegir mi camino.

Contents

1. Introduction

All models are wrong, but some are useful

— *George Box*

This thesis deals with the mathematical models used to forecast future asset prices. Estimating asset prices is arguably one of the most relevant problems for risk managers, central bankers, and investors. Traditional statistical methods rely on point estimates or confidence intervals to estimate future realizations. However, when it comes to analyzing asset prices at a future date, obtaining the full price distribution significantly improves the information available for decision-making. This is particularly relevant in financial prices, which typically exhibit asymmetries, fat tails and other non-normal features. Consequently, estimation methods relying on mean-variance approximations do not appropriately reproduce the real-world characteristics of financial asset prices, leading to biased predictions and inappropriate model choices.

However, while the rationale to move beyond a Gaussian framework is clear, there is a remarkable lack of consensus on which mathematical models are best suited to estimate future asset prices. The disagreement remains despite the continuous improvements and extensive use that price evolution models have experienced over the last decades. Although different approaches have been proposed to forecast future asset prices, the lack of consensus is driven by three main reasons:

First, estimating and calibrating the mathematical models used to forecast future asset prices has become increasingly complex, posing technical challenges that often lead to mispricing. Biased prices can invalidate the outputs of asset pricing models, leading to probabilistic paths that are not representative of real-world dynamics. Given these challenges, an appropriate implementation of asset evolution models requires that all mathematical nuisances and limitations of each forecast scheme are properly understood and considered in the implementation process. Technical challenges are particularly relevant in the stochastic processes that have been developed in the last decades, where the new features included to better approximate the asset price dynamics complicate the estimation and calibration process.

Second, within the mathematical finance community, most stakeholders already hold notably dogmatic views about the types of models that should or should not be used in asset price forecasting. However, these views are generally supported by subjective beliefs about the relative merits of each competing approach instead of proper empirical validation. In the academic literature, comprehensive analyses of alternative price forecast models are scarce and mainly devoted to return and variance comparisons. In contrast, much fewer analyses consider the full price distribution, and generalization to other datasets is hindered by small sample problems, limited model choices, or non-holistic evaluations.

Third, beyond modelling aspects, there is also a need for an evaluation framework that consistently assesses the predictive power of probabilistic forecasts. Although different metrics have been proposed to evaluate specific angles of probabilistic forecasts, there is no framework that jointly considers such partial measures to obtain a global evaluation. The lack of a global framework complicates the comparison of alternative modeling approaches, leading to contradictory results in different studies and preventing consistent answers on which type of models perform better.

Consequently, to tackle the current challenges in estimating and evaluating density predictions for asset prices, our thesis considers three interrelated topics:

1. Our first topic concerns the analysis of the numerical difficulties that characterize the estimation and calibration of complex stochastic price processes. Since the seminal papers of Black–Scholes and Merton[1], processes where asset prices diffuse continuously have been extensively used in risk management and option pricing. However, to appropriately capture the asset price dynamics in the real-world, traditional diffusion models have been extended to encompass a variety of features including stochastic volatility, mean-reversion, or discontinuous jumps.

   While these refinements have increased the realism of stochastic asset processes, they also give rise to increasing complexity in the mathematical schemes that define future asset paths. In particular, evolution models which include stochastic volatility and/or jumps do not typically exhibit a tractable density that can be used to obtain the probability distribution at a future date T. Alternatively, the characteristic functions of complex stochastic processes are generally simpler and more tractable than their corresponding densities. Therefore, the use of Fourier transforms has rapidly gained traction and the pricing models developed in the last decades have mostly relied on characteristic functions to obtain option prices.

   However, the use of Fourier transforms also bring specific challenges due to discontinuities in the integrand functions, singularities at the lower or upper integration limits, or other mathematical nuisances. Furthermore, these challenges vary depending on the Fourier algorithm used to obtain option prices, the characteristic function employed to describe future asset prices, or even the parameter region considered in the calibration. As a result, specific integrals/summations routines can give rise to particular numerical problems, whereas instabilities can also arise under certain combinations of a Fourier-based method, characteristic function, and/or parameter region.

   Given these interdependencies, a better understanding of the different Fourier methods and their biases is paramount to avoid pricing errors and generate consistent price forecasts.

2. Our second topic concerns the statistical comparison of the main probabilistic models used to forecast future asset prices. Among forecast models, the most commonly used schemes to estimate future asset prices are:

   - **Historical-based predictions:** Historical methods generate future predictions based on past prices. These models are easy to implement and extensively used in financial economics. However, it is well-known that historical patterns do not repeat themselves, particularly in times of economic turmoil. Furthermore, historical models may yield different estimates depending on the length of the calibration window, introducing uncertainty and cherry-picking concerns.

[1] See Black and Scholes (1973) and Merton (1973).

- **Risk-neutral methods:** Risk-neutral estimates contain forward-looking expectations and react immediately to changing market conditions, thus being conceptually better suited for forecasting purposes. However, risk-neutral models do not consider investors' risk preferences across different wealth states.

- **Risk-adjusted models:** By incorporating how investors value monetary outcomes in different states, these models provide a more realistic description of how investors operate in the real-world. However, risk-adjusted models are bounded by the assumption that investors are perfectly rational and always act without bias in their investment decision.

- **Sentiment-adjusted forecasts:** Since the pioneering work of Keynes (1936), increasing evidence shows that investors commit systematic behavioral mistakes that manifest in asset prices. If we accept that market prices can be affected by sentiment, it follows that market-implied forecasts should be appropriately adjusted to disentangle investor biases from fundamental expectations.

Despite their differences, all these models are still extensively used for forecasting purposes. Historical models are mainly used by risk managers; risk-neutral models are used for asset pricing; risk-adjusted models are common in economics, and sentiment-based predictions are used in behavioral finance. The reason why all these models are still used is the disagreement across different stakeholders on the relative merits and drawbacks of each alternative approach.

Conceptually, by including up-to-date expectations and higher realism in the processes used to describe market conditions and investors' behavior, the forecasting ability should improve, leading to better predictions as we move from historical-based models to sentiment-adjusted predictions. However, comparisons across the different modelling approaches are scarce and when it comes to evaluating entire probability distributions there are no empirical analyses that comprehensively assess the information content of the competing schemes.

3. Our third topic concerns the framework to evaluate probabilistic forecasts. Currently, there are several statistical tests and scoring rules that are designed to measure specific angles of forecast performance (e.g.: statistical consistency, local accuracy, global errors, etc.). However, since different metrics can lead to diverging model choices, there is a need for a comprehensive evaluation framework that jointly considers the different partial aspects of probabilistic forecasts. The lack of a common framework further complicates the evaluation of competing forecasts, as partial evaluations can give rise to contradictory results and model scores that have been obtained under incomplete information.

2. Publications derived from this thesis

This thesis is presented by a compendium of publications. The research articles included in this thesis are all published in academic journals that are indexed in the Journal Citation Report (JCR).

The full references of the articles that have been derived from this thesis are the following:

1. Crisóstomo, R. (2018). Speed and biases of Fourier-based pricing choices: a numerical analysis. *International Journal of Computer Mathematics*, 95:8, 1565-1582. https://doi.org/10.1080/00207160.2017.1322691
2. Crisóstomo, R. and Couso, L. (2018). Financial density forecasts: A comprehensive comparison of risk-neutral and historical schemes. *Journal of Forecasting* 37: 589-603. https://doi.org/10.1002/for.2521
3. Crisóstomo, R. (2021). Estimating real-world probabilities: A forward-looking behavioral framework. *Journal of Futures Markets*, 41, 1797-1823. https://doi.org/10.1002/fut.22248

A summary of each article is included in Chapter 5. In terms of quality indexes, the report showing the Journal Impact Factor (JIF), the JIF percentile, and the corresponding JCR quartile of each academic journal are presented in Chapter 7.

3. Hypothesis and objectives

To address the challenges stated in Chapter 1, we formulate the following research objectives and hypotheses. The academic paper/s in which each objective is addressed, and the corresponding hypothesis is rejected or validated are detailed after each objective:

Objective 1

> *To analyze the pricing biases and computational efficiency of the most commonly used Fourier-based methods for option pricing.*

Hypothesis 1

> *Different Fourier-based techniques exhibit different truncation, discretization, and interpolation errors. The pricing biases vary not only across alternative Fourier methods but also depend on the functional form of the characteristic function used to describe the asset evolution or even the parameter region. Beyond computational efficiency, these biases can lead to some methods failing to provide feasible option prices for specific stochastic models and/or parameter regions.*

This objective is addressed in our paper Crisóstomo, R (2018): **Speed and biases of Fourier-based pricing choices: a numerical analysis**, International Journal of Computer Mathematics, 95:8, 1565-1582, DOI: 10.1080/00207160.2017.1322691

Crisóstomo (2018) analyzes the truncation, discretization, and interpolation errors of seven Fourier methods for option pricing. We find that both truncation and discretization errors increase as we move from the classic Black-Scholes-Merton (BSM) model to more complex stochastic processes. Overall, we find that the COS method developed by Fang and Oosterlee (2009) is the fastest and most accurate across a range of underlying stochastic processes, strike regions, and the number of options priced. We also find that for the Variance Gamma process, most Fourier methods fail to provide feasible option prices under a challenging parameter region.

Objective 2

> *To develop a pricing algorithm that improves the traditional FFT, increasing both its accuracy and computational time*

Hypothesis 2

> *The FFT was a notable improvement in computational option pricing in 1999, but new algorithms can be used to improve the FFT performance in terms of both speed and accuracy.*

This objective is addressed in Crisóstomo (2018): Speed and biases of Fourier-based pricing choices: a numerical analysis. This paper shows that the Carr-Madan formula, which is the precursor of the FFT, can be optimized through strike vectorizations in a simple way which simultaneously improves the speed and the accuracy of the traditional FFT. Compared to the FFT, the flexibility of the strike-optimized Carr-Madan formula (CM-OPT) allows: (i) pricing any required strikes; (ii) choosing any integration size and technique; and (iii) avoiding interpolation biases. As a result, the CM-OPT is both faster and more accurate than the traditional FFT, rendering this method inefficient.

Objective 3

> *To analyze the forecasting ability of the most commonly used historical and risk-neutral methods to obtain probabilistic predictions for future asset prices.*

Hypothesis 3

> *Risk-neutral forecasts obtained from option prices, given their forward-looking nature, outperform historical-based predictions in information content.*

This objective is addressed in our paper Crisóstomo, R and Couso, L (2018): **Financial density forecasts: A comprehensive comparison of risk-neutral and historical schemes.** *Journal of Forecasting* 37: 589-603. https://doi.org/10.1002/for.2521

Despite its extensive use in quantitative finance, there are no comprehensive analyses of the relative merits and drawbacks of risk-neutral versus historical-based predictions in probabilistic forecasting. Using a data sample of over 21 years, Crisóstomo and Couso (2018) assess the statistical consistency, local accuracy, and forecasting errors of a wide range of forecast models, showing that risk-neutral methods outperform historical-based predictions in terms of information content.

Objective 4

> *To develop an evaluation framework that generates a consistent ranking of probabilistic predictions on a common scale.*

Hypothesis 4

> *Different angles of probabilistic forecasting can be summarized in a comprehensive measure that aggregates the local accuracy, global errors, and statistical consistency of forecast schemes in a standardized score.*

This objective is addressed in our published paper Crisóstomo and Couso (2018): Financial density forecasts: A comprehensive comparison of risk-neutral and historical schemes. *Journal of Forecasting* 37: 589-603. https://doi.org/10.1002/for.2521

Our paper develops a new scoring system that integrates the results from the statistical consistency, local accuracy, and forecasting error analyses in a single ranking. To aggregate the partial measures, we first normalize the outcomes obtained in the three partial categories into standardized scales. Next, the Integrated Forecast Score (IFS) is constructed by aggregating the normalized statistical consistency, local accuracy, and forecasting errors scores in a joint [0, 1] scale.

Objective 5

> *To develop a quantitative framework to measure investor sentiment and quantify its effects on probabilistic forecasts.*

Hypothesis 5

> *Investor sentiment is strongly time-variant and manifests separately in different areas of the return distribution. Sentiment-based biases affect the mean, the dispersion, and the weights that investors assign to the tails of the distribution.*

This objective is addressed in our paper Crisóstomo, R. (2021). **Estimating real-world probabilities: A forward-looking behavioral framework**. *Journal of Futures Markets* 41, 1797–1823.
https://doi.org/10.1002/fut.22248

Crisóstomo (2021) develops a novel framework to measure investor sentiment and quantify its effects. Our framework considers the joint effect of three investor biases: excessive optimism, which generates biases on average returns; overconfidence, which impacts volatility predictions, and tail sentiment, which is related to non-rational tail expectations. All investor biases are then aggregated into an ex-ante behavioral stochastic discount factor (SDF) which is used to transform classical forecasts into real-world predictions.

Our results show that, for a wide sample of stochastic models and risk-preference combinations, a simple behavioral correction generates substantial forecast gains. The information improvement is robust across all evaluation methods, risk preferences, and sentiment calibration.

Objective 6

> *To develop a trading strategy that exploits behavioral biases to generate market profits.*

Hypothesis 6

> *The statistical improvement of our behavioral framework can be exploited in practice through an options trading strategy that generates trading profits in the stock market.*

This objective is addressed in our paper Crisóstomo (2021): Estimating real-world probabilities: A forward-looking behavioral framework. *Journal of Futures Markets* 41, 1797–1823.
https://doi.org/10.1002/fut.22248

Crisóstomo (2021) develops a trading strategy that exploits the misspecifications of traditional densities. Our strategy is designed to benefit from sentiment-induced biases by going long states with too-low probability and short states with too-high probability. Using an option-based strategy, we show that excessive sentiment can be exploited to generate market-based returns that are almost twice the expected return under the utility-adjusted benchmark, and 28 times higher than the return of a risk-free investment. Remarkably, all our sentiment strategies (i.e., those concerning optimism, overconfidence, and tail sentiment) contribute to trading gains.

Objective 7

> *To test the improvements of our forward-looking sentiment framework against a statistical recalibration that corrects density predictions in light of past mistakes.*

Hypothesis 7

> *A forward-looking sentiment framework achieves better results than using backward-looking biases to correct density predictions.*

This objective is addressed in our paper Crisóstomo (2021). Estimating real-world probabilities: A forward-looking behavioral framework. *Journal of Futures Markets* 41, 1797–1823.
https://doi.org/10.1002/fut.22248

The average biases observed in past predictions can be corrected through a recalibration of the current density forecast in light of past mistakes. By construction, the recalibrated density corrects all the forecast biases observed in past predictions, hence including both risk preferences and sentiment-induced mitakes. Through a comparison of our sentiment framework with a standard method to correct past mistakes, we show that our forward-looking framework outperforms the use of historical information to adjust probabilistic forecasts.

4 Theoretical framework and methodology

This thesis analyzes the different alternatives used to forecast future asset prices. In terms of methodological tools, we employ the following quantitative methods:

4.1 Fourier transforms for option pricing

Under the no-arbitrage paradigm, option prices are calculated as the present value of the expected option payoff under the risk-neutral measure

$$V_0 = e^{-rT} E_Q \left[H(S_t) \right] \quad \textbf{(1)}$$

where V_0 is the option value at time t = 0, S_t the underlying price, r the risk-free rate, T the time to maturity, $H(S_t)$ is the option payoff and $E_Q[\bullet]$ denotes the expectation operator under the risk-neutral measure. When the underlying asset's density function is available, the present value of a European call with strike K and expiration T is given by $H(S_t) = (S_T - K)^+$. Thus, its present value at time t = 0 can be obtained as

$$C(T,K) = e^{-rT} \int_0^\infty (S_T - K)^+ q(S_T) dS_T \quad \textbf{(2)}$$

Where $q(S_T)$ is the risk-neutral density of the underlying asset S_t at the terminal date T. However, numerous asset processes do not exhibit a tractable density. Alternatively, by expanding (2), it is straightforward to show that the price of a European call can be expressed as

$$C(T,K) = S_0 \, \Pi_1 - e^{-rT} K \, \Pi_2 \quad \textbf{(3)}$$

where Π_1 and Π_2 are two probability-related quantities. Specifically, Π_1 is the option delta while Π_2 is the risk-neutral probability of exercise $\mathrm{P}(S_T > K)$. For processes that do not exhibit a tractable density, Bakshi and Madan (2000) show that these probabilities can be computed as

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left[\frac{e^{-iw\ln(K)} \psi_{\ln S_T}(w-i)}{iw\psi_{\ln S_T}(-i)} \right] dw \quad \textbf{(4)}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left[\frac{e^{-iw\ln(K)} \psi_{\ln S_T}(w)}{iw} \right] dw \quad \textbf{(5)}$$

Where $\psi_{\ln S_T}$ is the characteristic function of the log-asset price[2] and $\mathrm{Re}[\bullet]$ denotes the real operator. To increase the computational efficiency of the delta-probability decomposition (DPD), Zhu (2010) proposes to introduce a vector of strikes $\mathbf{K}$ in the calculation of (4) and (5). The probability vectors $\mathbf{\Pi_1}$ and $\mathbf{\Pi_2}$ are hence given by

[2] The characteristic function is the Fourier transform of the probability density function.

$$\mathbf{\Pi_1} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iw\ln(\mathbf{K})}\psi_{\ln S_T}(w-i)}{iw\psi_{\ln S_T}(-i)}\right]dw \tag{6}$$

$$\mathbf{\Pi_2} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iw\ln(\mathbf{K})}\psi_{\ln S_T}(w)}{iw}\right]dw \tag{7}$$

And, thus, the vector of call prices can be computed as

$$\mathbf{C}(T,\mathbf{K}) = S_0\ \mathbf{\Pi_1} - e^{-rT}\mathbf{K}\ \mathbf{\Pi_2} \tag{8}$$

Similarly, to improve computational efficiency, Attari (2004) proposes a DPD reformulation that merges integrands (4) and (5) into a single pricing expression of the form

$$C(T,K) = S_0 - e^{-rT}K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty I_A(w)dw\right) \tag{9}$$

where

$$I_A(w) = \frac{(\mathrm{Re}(\psi_{\ln S_T}(w)) + \frac{\mathrm{Im}(\psi_{\ln S_T}(w))}{w})\cos(w\ln(K)) + (\mathrm{Im}(\psi_{\ln S_T}(w)) - \frac{\mathrm{Re}(\psi_{\ln S_T}(w))}{w})\sin(w\ln(K))}{1+w^2} \tag{10}$$

Alternatively, to harness the power of the fast Fourier transform (FFT) in option pricing, Carr and Madan (1999) developed an algorithm that exploits periodicities and symmetries in the characteristic function. Since the FFT can only be used in square-integrable functions, Carr-Madan's approach considers a modified call price where a dampening factor $e^{\alpha\ln(K)}$ is introduced to avoid the divergence at $w = 0$

$$C_{\mathrm{mod}}(T,K) = e^{\alpha\ln(K)}C(T,K) \tag{11}$$

Using Fourier inversions, Carr-Madan's paper shows that the original call price can be recovered as:

$$C(T,K) = \frac{e^{-\alpha\ln(K)-rt}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iw\ln(K)}\psi_{\ln S_T}(w-(\alpha+1)i)}{\alpha^2+\alpha-w^2+i(2\alpha+1)w}\right]dw \tag{12}$$

Although (12) can be directly used to compute call prices, it is common to evaluate it through the FFT. Specifically, by setting the grid points as $w_n = (n-1)\Delta w$ and using the trapezoidal rule, the price of N European call options can be computed as

$$\widehat{C}(k_m) \approx \sum_{n=1}^{N} e^{-i\frac{2\pi}{N}(n-1)(m-1)} g(w_n) \quad for\, m = 1, ..., N \tag{13}$$

where

$$g(w_n) = e^{ibw_n + \alpha k_m - rT} \frac{\psi_{\ln S_T}(w_n - (\alpha+1)i)}{\alpha^2 + \alpha - w_n{}^2 + i(2\alpha+1)w} \Delta w \tag{14}$$

More recently, Fang and Oosterlee (2009) introduced a pricing method that relies on Fourier-cosine expansions and offers an efficient way to recover the density of the underlying from the characteristic function. Specifically, by: i) expressing Equation (2) in terms of ln(St/K) in the truncated interval [a,b]; ii) replacing the density and option payoff by the first N terms of their Fourier-cosine expansion; and iii) approximating the density-related coefficients using their characteristic function representation, the price of a European call can be obtained as:

$$C(T,x) \approx e^{-rT} \sum_{n=0}^{N-1}{}' \operatorname{Re}\left\{ \psi\left(\frac{n\pi}{b-a}\right) e^{in\pi\frac{x-a}{b-a}} \right\} V_n \tag{15}$$

with

$$V_n = \frac{2}{b-a} \int_a^b v(y,T) \cos\left(n\pi \tfrac{y-a}{b-a}\right) dy \tag{16}$$

where $v(y,T)$ is the option payoff, $x = \ln(S_0 / K)$, $y = \ln(S_T / K)$ and $\sum{}'$ indicates that the first summation term is weighted by one-half.

Following these methods, Crisóstomo (2018) investigates the pricing biases and computational speed of seven Fourier-based pricing choices:

- **DPD:** Delta-probability decomposition. Call values are individually computed through Equations (3)–(5)
- **DPD-OPT:** Optimized DPD. Strike vector computations are used to simultaneously compute call values for a variety of strikes. Equations (6)–(8) are used.
- **AT-OPT**: Optimized Attari approach. Call values are computed with Equations (9) and (10). The CPU burden is optimized through strike vectorizations.
- **COS-OPT:** Optimized COS method. A multi-strike version of Equations (15) and (16) is used to calculate option prices. Following Fang and Oosterlee, the truncation range is obtained through the first four cumulants of ln(ST/K) and a scale parameter L = 3 is employed.
- **FFT:** Standard FFT. Vector operations (instead of loops) are used to improve the performance. After experimenting with different values, we settle for an α = 1.75, which delivers a 10^{-10} accuracy for all the models tested. Options that do not exactly fall in the FFT strike grid are exponentially interpolated.

- **FFT-SA:** Strike-adjusted FFT. Call values are determined by successive FFT runs. Strike grids are adjusted to match all the required options in at least one FFT run, thus avoiding interpolation.
- **CM-OPT:** Optimized Carr–Madan formula. Call values are computed using Equation (12) and strike vector computations.

4.2 Forecasting methods to obtain asset price densities

Our thesis makes extensive use of the mathematical methods employed to forecast future asset prices. In particular, the most commonly used benchmarks to obtain density forecasts are historical-based methods and risk-neutral predictions.

4.2.1 Historical-based predictions

Regarding historical methods, our thesis considers five benchmark specifications. The first assumes that future prices follow a geometric Brownian motion and thus the corresponding price density is lognormally distributed. Our second specification generates future price paths by a bootstrapping of past returns. For each observation date t, the one-day-ahead return is given by:

$$r_{t+1} = \mu + z_{t+1}\,, \quad z_{t+1} \sim \left\{ r^h \right\} \tag{17}$$

where $\left\{ r^h \right\} = (r_1^h, ..., r_t^h)$ denotes the set of historical returns and μ is the daily average return. Next, we consider two standard GARCH(1,1) models, where returns are given by

$$\begin{aligned} r_{t+1} &= \mu + e_{t+1} \\ e_{t+1} &= \sigma_{t+1} z_{t+1}, \quad z_{t+1} \sim f_p(0,1) \\ \sigma_{t+1}^2 &= \omega + \alpha e_t^2 + \beta \sigma_t^2 \end{aligned} \tag{18}$$

and the standardized residuals z_t are obtained from either a Gaussian (GARCH-N) or a Student's t distribution (GARCH-t).

In addition, we evaluate the filtered historical simulation (FHS) approach introduced in Barone-Adesi, Engle, and Mancini (2008), which combines an asymmetric GJR-GARCH model of Glosten, Jagannathan, and Runkle (1993) with empirical innovations. Specifically, future returns in the GJR-FHS(1,1) model are computed as

$$\begin{aligned} r_{t+1} &= \mu + e_{t+1} \\ e_{t+1} &= \sigma_{t+1} z_{t+1}, \quad z_{t+1} \sim f_{np}(0,1) \\ \sigma_{t+1}^2 &= \omega + \alpha e_t^2 + \beta \sigma_t^2 + \gamma I_t e_t^2 \end{aligned} \tag{19}$$

Where I_t = 1 when e_t < 0 and 0 otherwise, introducing a leverage effect.

4.2.2 Risk-neutral methods

Regarding risk-neutral specifications, we employ five benchmark density models. Our simplest model is again a lognormal specification but calibrated to match option prices at time t. We next consider the Heston (1993) model, which employs mean-reverting stochastic volatility with risk-neutral dynamics given by

$$dF_t = \sqrt{V_t}F_t dW_{t,1} \quad \textbf{(20)}$$

$$dV_t = a(\bar{V} - V_t)dt + \eta\sqrt{V_t}dW_{t,2} \quad \textbf{(21)}$$

where $dW_{t,1}$ and $dW_{t,2}$ are two correlated Wiener processes. Following Bates (1996), we also complement the Heston volatility in (17) with a lognormal price jump, thus obtaining the dynamics:

$$dF_t = \sqrt{V_t}F_t dW_{t,1} + J_t F_t dN_t - \lambda\mu_J F_t dt \quad \textbf{(22)}$$

where N_t is a Poisson process with intensity λ and J_t are the jumps sizes, which are lognormally distributed with an average size μ_J and standard deviation v_J.

Leaving diffusion, we also evaluate the purely discontinuous Variance Gamma (VG) model (Madan, Carr, and Chang, 1998), which combines frequent small moves with rare big jumps. The VG dynamics are:

$$\begin{aligned} F_{t^*} &= F_t e^{\lambda\tau + H(\tau;\sigma,v,\theta)} \\ \lambda &= \frac{1}{v}\ln(1-\theta v - \frac{\sigma^2 v}{2}) \\ H(\tau;\sigma,v,\theta) &= \theta G(\tau;v) + \sigma G(\tau;v)W_t \end{aligned} \quad \textbf{(23)}$$

where $G(\tau;v)$ is a Gamma distribution and the parameters σ, v and θ jointly control the volatility, asymmetry, and kurtosis.

Finally, Breeden and Litzenberger (1978) show that given a continuous of non-arbitrable call prices, it is possible to obtain a unique risk-neutral distribution that replicates exactly such option prices. Specifically, we employ the Malz (2014) implementation. For each expiry t^*, the interpolated volatility function is used to compute the continuous call pricing function $C(x,t^*)$, and these prices are then numerically differentiated to obtain the CDF for all the strikes x as:

$$CDF_{t^*}(x) \approx 1 + e^{r\tau}\frac{1}{\Delta}\left[C(x-\frac{\Delta}{2},t^*) - C(x+\frac{\Delta}{2},t^*)\right] \quad \textbf{(24)}$$

where Δ denotes the step size used in the finite differentiation.

4.3 Density forecasts verification

Evaluating financial densities is not straightforward, as only one realization is available to evaluate each entire density prediction. This issue can be tackled by working with ensemble predictions, thus jointly assessing a sequence of predictive densities and the corresponding sequence of realizations. Diebold, Gunther, and Tay (1998) show that the statistical consistency between a sequence of probabilistic forecasts and the corresponding realizations can be assessed through PIT analyses. For a given date t, the PIT represents the quantile of the *ex-ante* distribution at which the *ex-post* realization is observed. Thus,

$$PIT_t = \int_{-\infty}^{x_{t^*}} f_t(x)\,dx \qquad \textbf{(25)}$$

In a well-specified model, the observed realizations should be indistinguishable from random draws from the predictive distributions, and therefore the sequence of PIT values should be uniformly distributed in the (0, 1) range. Given the low power of uniformly distributed tests, Berkowitz (2001) proposes a reformulation of the PIT values into a transformed sequence (T-PIT) that should be formed by i.i.d. $N(0,1)$ variables in a correctly specified density model[3]. The Berkowitz test first computes the T-PIT values as $T\text{-}PIT_t = \Phi^{-1}(PIT_t)$, and next the AR(1) model

$$T\text{-}PIT_t - \mu = \rho(T\text{-}PIT_{t-1} - \mu) + \varepsilon_t \qquad \textbf{(26)}$$

is estimated to assess the mean, variance, and serial correlation through the likelihood ratio test LR3 = $-2(L(0,1,0) \; - \; L(\hat{\mu},\; \hat{\sigma}^2, \hat{\rho}))$.

Additionally, to complement PIT-based tests, the accuracy of different forecasting schemes can be compared through the likelihood of the *ex-post* realizations evaluated with the *ex-ante* distribution as

$$L = \sum_{t=1}^{N} \log(f_t(x_{t^*})) \qquad \textbf{(27)}$$

where f_t denotes the *ex-ante* density computed at observation date t and x_{t^*} denotes the *ex-post* realization at time t^*. See Liu et al. (2007).

Although the logarithmic score considers the likelihood of the *ex-post* realizations, thus measuring the local accuracy of a forecasting scheme, it ignores any other probability masses. In contrast, the CRPS weights the entire ex-ante distribution, measuring the statistical distance between the actual realization and all other probabilistic outcomes (Matheson and Winkler (1976)). Denoting by CDF^m and CDF^r the cumulative distributions of the forecasting model and the realization, the CRPS is given by:

$$CRPS_t = \int_{-\infty}^{\infty} \left(CDF^m(x) - CDF^r(x)\right)^2 dx \qquad \textbf{(28)}$$

[3] This reformulation brings about the more powerful tests associated with Gaussian variables.

where:

$$CDF^{r}(x)=\begin{cases}0 & for\ x_{t^*}<x\\ 1 & for\ x_{t^*}\geq x\end{cases} \quad \textbf{(29)}$$

Consequently, the CRPS analyzes the global errors of a forecasting scheme, thus complementing the local accuracy metric obtained from the log score.

4.4 Risk preferences, sentiment, and the stochastic discount factor

Our thesis uses the Stochastic Discount Factor (SDF) framework to analyze and incorporate the impact of risk preferences and sentiment in density predictions. Following Cochrane (2005), the present value of any asset P_t can be computed as the expectation of future cash flows, discounted by the SDF.

$$P_t = E_t\left[m_T z_T\right] \quad \textbf{(30)}$$

where E_t is the expectation under the representative investor measure, z_T are the state-dependent payoffs, and m_T is the pricing kernel or SDF, which summarizes investors' preferences and beliefs. Expressed in integral form, the pricing equation becomes:

$$P_t = \int_{\mathbb{R}} m_t(x_T)\, z_T(x_T)\, f_t(x_T)\, dx_T \quad \textbf{(31)}$$

where $f_t(x_T)$ represents the probability density function of future asset prices x_T. In a risk-neutral world, investors' preferences reflect time-discount only. Therefore, the pricing kernel is simply $m_T^{\mathbb{Q}} = e^{-r\tau}$, and by substituting in (31), the value of any asset becomes:

$$P_t = e^{-r\tau}\int_{\mathbb{R}} z_T(x_T)\, f_t^{\mathbb{Q}}(x_T)\, dx_T \quad \textbf{(32)}$$

where $f_t^{\mathbb{Q}}(x_T)$ represents the risk-neutral density function calculated at t and with forecast horizon T.

4.4.1 Risk-adjusted densities

Regarding risk preferences, rational investors exhibit risk aversion, attaching a decreasing marginal utility to payoffs received in states of higher wealth. Liu et al. (2007) show that the pricing kernel for risk-averse investors is proportional to its marginal utility and given by

$$m_T^{RW} = e^{-r\tau} u'(x_T)\Psi(x_T) \quad \textbf{(33)}$$

Therefore, the pricing equation becomes:

$$P_t = e^{-r\tau}\int_{\mathbb{R}} z_T(x_T)\, f_t^{RA}(x_T) u'(x_T) dx_T \quad \textbf{(34)}$$

Comparing (32) and (34), it follows that for the risk-neutral and risk-adjusted economies to generate the same market price, the risk-adjusted density f_t^{RA} should be equal to $f_t^{\mathbb{Q}}/u'$. However, Bliss and Panigirtzoglou (2004) note that the transformation from risk-neutral to risk-adjusted probabilities exhibits non-linearities, and hence the scaling factor $\int_o^{\infty} f_t^{\mathbb{Q}}(y)/u'(y)dy$ should be employed to ensure integration to unity. Consequently, starting from any RND, the risk-adjusted probabilities can be obtained as:

$$f_t^{RA}(x_T) = \frac{f_t^{\mathbb{Q}}(x_T)/u'(x_T)}{\int_o^{\infty} f_t^{\mathbb{Q}}(y)/u'(y)dy} \quad \textbf{(35)}$$

For example, for investors featuring a power utility with constant relative risk aversion (CRRA),

$$u^{CRRA}(x_T) = \begin{cases} \frac{1}{1-\gamma} x_T^{1-\gamma} & if \ \ \gamma \geq 0; \gamma \neq 1 \\ \ln x_T & if \ \ \gamma = 1 \end{cases} \quad \textbf{(36)}$$

where γ denotes the coefficient of relative risk aversion. The marginal CRRA utility is then $u' = x_T^{-\gamma}$, and the pricing kernel becomes $m_T^{CRRA} = e^{-r\tau} x_T^{-\gamma}$. Therefore, the risk-adjusted density f_t^{CRRA} is:

$$f_t^{CRRA}(x_T) = \frac{x_T^{\gamma} f_t^{\mathbb{Q}}(x_T)}{\int_o^{\infty} y^{\gamma} f_t^{\mathbb{Q}}(y)dy} \quad \textbf{(37)}$$

4.4.2 Sentiment-based predictions

The effect of investor sentiment in density predictions can be also expressed through the SDF. To examine how behavioral effects impact different areas of the return distribution, we aggregate three sentiment-induced mistakes: excessive optimism, which relates to biases in average returns; overconfidence, which leads to errors in volatility predictions, and tail sentiment, which is linked to non-rational tail expectations.

The sentiment function $\Psi(x_T)$ summarizes the behavioral corrections required to transform risk-adjusted forecasts into real-world densities. In terms of the SDF, the aggregate effect of investor optimism, investor overconfidence, and tail sentiment is obtained as:

$$\Psi(x_T) = m_T^{mv} m_T^{ts} \quad \textbf{(38)}$$

Where m_T^{mv} is the mean-variance pricing kernel and m_T^{ts} is the tail-shift pricing kernel. Consequently, the real-world pricing kernel m_T^{RW}, which reflects the cumulative impact of investor sentiment and investor risk preferences is given by:

$$m_T^{RW} = e^{-r\tau} u'(x_T) \Psi(x_T) \quad \textbf{(39)}$$

Following equations (31) to (35), the real-world density can be obtained from an initial RND $f_t^{\mathbb{Q}}$, marginal utility u', and sentiment function Ψ as:

$$f_t^{RW}(x_T) = \frac{f_t^{\mathbb{Q}}(x_T) / u'(x_T)\Psi(x_T)}{\int_o^{\infty} f_t^{\mathbb{Q}}(y) / u'(y)\Psi(y)dy} \quad \textbf{(40)}$$

More specifically, our framework obtains density predictions in terms of the behavioral tuple $\{\theta_1, \theta_2, \theta_3\}$. For any density forecast $f_t\ (x_T)$ the correction transformation due to investor optimism and overconfidence can be obtained through a linear mapping of the original values x_T into the adjusted values $\widehat{x}_T$

$$\widehat{x}_T = \theta_{1,t} + x_T\theta_{2,t} + (1-\theta_{2,t})\mu_{X_T} \quad \textbf{(41)}$$

where θ_1 and θ_2 denote the location and scale shift parameters, respectively. This transformation shifts the mean and standard deviation of the traditional forecast into the adjusted values $\widehat{\mu} = \mu + \theta_1$ and $\hat{\sigma} = \sigma\,\theta_2$. After the behavioral transformation, the mean-variance pricing kernel m_T^{mv} is obtained as:

$$m_T^{mv} = \frac{f_t^{RA}(x_T)}{f_t^{mv}(x_T)} \quad \textbf{(42)}$$

where $f_t^{RA}(x_T)$ represents the risk-adjusted density and $f_t^{mv}(x_T)$ is the behavioral distribution obtained through the mean-variance shift.

To account for biases in tail expectations, we employ an adjustment that progressively shifts probability mass from the left to the right tail or vice versa. Denoting by $q(\alpha)$ and $q(1-\alpha)$ the quantiles which define the left and right tails respectively, we obtain the tail-shift pricing kernel m_T^{ts} as

$$m_T^{ts} = \begin{cases} e^{\theta_3(q(\alpha)-x_T)} & for\ x_T \in (-\infty, q(\alpha)) \\ 1 & for\ x_T \in [q(\alpha), q(1-\alpha)] \\ e^{-\theta_3(x_T - q(1-\alpha))} & for\ x_T \in (q(1-\alpha), \infty) \end{cases} \quad \textbf{(43)}$$

where θ_3 controls the direction and intensity of the tail-shifting. The log-linearity of m_T^{ts} ensures that all tail-adjusted probabilities remain positive even for extreme values of the density domain.

5 Summary of the published papers

5.1 Article 1: Crisóstomo, R. (2018). Speed and biases of Fourier-based pricing choices: a numerical analysis. *International Journal of Computer Mathematics*, 95:8, 1565-1582. https://doi.org/10.1080/00207160.2017.1322691

ABSTRACT

We compare the CPU effort and pricing biases of seven Fourier-based implementations. Our analyses show that truncation and discretization errors significantly increase as we move away from the Black–Scholes–Merton framework. We rank the speed and accuracy of the competing choices, showing which methods require smaller truncation ranges and which are the most efficient in terms of sampling densities. While all implementations converge well in the Bates jump-diffusion model, Attari's formula is the only Fourier-based method that does not blow up for any Variance Gamma parameter values. In terms of speed, the use of strike vector computations significantly improves the computational burden, rendering both fast Fourier transforms (FFT) and plain delta-probability decompositions inefficient. We conclude that the multi-strike version of the COS method is notably faster than any other implementation, whereas the strike-optimized Carr Madan's formula is simultaneously faster and more accurate than the FFT, thus questioning its use.

INTRODUCTION

Since the seminal papers of Black–Scholes and Merton, processes where asset prices diffusecontinuously have been extensively used in risk management and option pricing. Diffusion models exhibit a variety of forms, including stochastic volatility, mean-reversion or seasonality, and their widespread use highlights the success that these models have achieved in financial modelling. Yet casual observation reveals that the prices of traded assets routinely undergo jumps. Discontinuities can occur, for instance, due to unexpected news, due to trading restrictions or simply because there is a substantial imbalance between buy and sell orders.

The importance of jump modelling becomes evident if we analyze the prices of short-dated out-of-the-money (OTM) options. The value of these contracts critically stems from an expectation of large underlying movements. However, empirical studies have shown that diffusion-only models cannot consistently generate the asymmetry and fat-tails that are routinely implied by short-term OTM options.

This paper contributes to the option pricing literature by benchmarking the speed and accuracy of seven Fourier-based pricing choices. Specifically, our analyses focus on two jump models that have been proposed as a framework to price options with different strikes and maturities. First, the Bates jump-diffusion model, which blends the Heston dynamics with lognormally distributed price jumps. Second, the asymmetric variance gamma (VG), a purely discontinuous process where the underlying assets evolve through a combination of many small jumps and rare big moves.

Both models are implemented by means of characteristic functions. Fourier transforms are rapidly gaining traction in finance and most of the option pricing models developed in the last decade have relied on characteristic functions to obtain option prices. Thus, a better understanding of the different implementations is paramount to avoid pricing errors. We investigate the speed and biases of a wide range of Fourier pricing choices, including Delta-probability decompositions, the Carr–Madan and Attari formulae, the COS method, and fast Fourier transforms.

The novelty of our paper lies in:

1. We are the first to consider the strike-optimized version of the Carr–Madan and Attari formulas, and one of the first to benchmark the multi-strike version of the COS method. We show that all these alternatives significantly outperform the FFT.

2. We compare the numerical efficiency of seven Fourier-based alternatives, showing which methods require the highest/lowest integration range and the highest/lowest sampling densities.

3. We find that Attari's formula is the only method that does not blow up in any problematic region of the AVG model.

4. We show that the strike-optimized version of Carr–Madan's formula is simultaneously faster and more accurate than the FFT, questioning its widespread use.

An important reference in this respect is the BENCHOP competition. This project compares the accuracy and speed of several Fouriermethods, finding that the COS formula is the overall fastest alternative. To benchmark our results to this project, we employ the BENCHOP implementation for the COS method developed by Ruijter and Oosterlee, which we have adapted to simultaneously calculate option prices for different strikes.

CONCLUSIONS

Our paper analyses the speed and accuracy of seven Fourier-based pricing choices. We show that truncation errors increase as we move from the BSM to the Bates model and further intensify under the AVG dynamics. Discretization errors also increase when discontinuous jumps are considered, but the rise is modest and remains similar for both jump models.

Our analyses demonstrate the higher efficiency of strike vector computations compared to other traditional choices. In our tests, computing option prices through the AT-OPT, CM-OPT, DPD-OPT and COS-OPT is, on average, 54, 67, 165 and over 1500 times faster than in the FFT. We show that the multi-strike version of the COS formula is the overall fastest alternative, a result that stems from the lower truncation range required in the COS method and the rapid decay of the cosine series coefficients.

We find that among quadrature-based methods: (i) the DPD-OPT exhibits the highest sampling efficiency but also the slowest decay rate, (ii) the CM-OPT stands out for minimizing truncation errors in the Fourier space and (iii) the AT-OPT suffers the largest discretization errors, requiring higher values of N to achieve the same level of accuracy. As a result, the DPD-OPT performs best when pricing a high number of options, the CM-OPT is more efficient when only a few prices are required, while the AT-OPT typically ranks as the slowest strike-optimized alternative.

We show that obtaining accurate option values can be particularly challenging in the AVG model. While all methods converge well under the BSM and Bates dynamics, large truncation errors significantly complicate the practical AVG implementation. Moreover, depending on the AVG parameters, specific Fourier implementations may completely fail to provide reasonable option prices: both the FFT and the CM-OPT can blow up in regions where inequality (30) is respected, whereas the DPD-OPT and COS-OPT also fail when (30) is not obeyed. In contrast, the AT-OPT seems to work fine for any AVG parameter values.

Finally, the comparison between the FFT and the CM-OPT deserves a special mention. While both are based on the same pricing approach, the CM-OPT's flexibility allows (i) pricing any required strikes, (ii) choosing any integration grid and (iii) avoiding interpolation biases. As a result, the CM-OPT is both faster and more accurate than the FFT, thus rendering this method inefficient. Based on our results, we see no reason to employ the FFT over the CM-OPT, but further analysis may be needed in order to confirm this hypothesis.

5.2 Article 2: Crisóstomo, R. and Couso, L. (2018). Financial density forecasts: A comprehensive comparison of risk-neutral and historical schemes. *Journal of Forecasting* 37: 589-603. https://doi.org/10.1002/for.2521

ABSTRACT

We investigate the forecasting ability of the most commonly used benchmarks in financial economics. We approach the usual caveats of probabilistic forecasts studies ─small samples, limited models, and nonholistic validations─ by performing a comprehensive comparison of 15 predictive schemes during a time period of over 21 years. All densities are evaluated in terms of their statistical consistency, local accuracy and forecasting errors. Using a new composite indicator, the integrated forecast score, we show that risk-neutral densities outperform historical-based predictions in terms of information content. We find that the variance gamma model generates the highest out-of-sample likelihood of observed prices and the lowest predictive errors, whereas the GARCH-based GJR-FHS delivers the most consistent forecasts across the entire density range. In contrast, lognormal densities, the Heston model, or the nonparametric Breeden-Litzenberger formula yield biased predictions and are rejected in statistical tests.

INTRODUCTION

Forecasting future asset prices is arguably one of the most relevant problems for risk managers, central bankers, and investors. Historical and risk‐neutral methods are the most widely used techniques in financial forecasting. Yet, when it comes to evaluate predictions across the entire density range, comprehensive comparisons are scarce and there is no consensus on which models provide better forecasts.

Historical methods generate future predictions based on past prices. These models are easy to implement and extensively used in risk management and stress testing. However, it is well-known that historical patterns do not repeat themselves, particularly in times of economic turmoil. Furthermore, historical models may yield different estimates depending on the length of the calibration window, introducing uncertainty and possible cherry-picking concerns.

Risk-neutral estimates, on the other hand, contain forward-looking expectations and react immediately to changing market conditions, thus being conceptually better suited for forecasting purposes. However, risk-neutral models do not explicitly consider the investors' risk preferencesacross different future states. Consequently, some agents rapidly dismiss risk-neutral models as the basis for financial predictions.

The previous literature on financial forecasts has been mainly devoted to volatility predictions. Much fewer studies consider entire density forecasts. While empirical analyses tend to find that risk-neutral densities (RNDs) outperform historical-based estimates, generalizations to other markets or time periods are typically limited by three methodological reasons. First, data availability issues have led most researchers to work with relatively small option samples. Limited samples can significantly impact the evaluation of predictive densities, as the inability to reject a particular model can be due to the low statistical power of the testing procedures.

Second, comparing density estimates from a wide range of schemes requires working with markedly different models and mathematical routines. As a result, most studies have contributed through vis-à-vis comparisons across particular model choices or by surveying specific asset dynamics. However, empirical analyses covering a comprehensive range of risk-neutral and historical densities are scarce.

Third, the validation of financial density forecasts is typically performed through the so-called probability integral transforms (PIT), which assess the statistical consistency between the ex ante densities and the ex post realizations. However, several papers have shown that PIT-based analyses do not consider the forecasting accuracy of the competing methods or the magnitude of its errors, advocating for targeted scoring rules to supplement the PIT assessments.

We approach these methodological caveats –small samples, limited models and nonholistic validations– by performing a comprehensive analysis of 15 forecasting schemes during a period of over 21 years. Historical densities are generated using a wide range of models, spanning from returns bootstrapping or standard GARCH dynamics to asymmetric models with filtered historical simulation. Similarly, we estimate RNDs using the most common benchmarks in financial economics, including lognormal densities, stochastic volatility, jump processes, and nonparametric distributions.

All density forecasts are evaluated through a threetiered criterion. First, we consider a multi-factor goodness-of-fit analysis, assessing each PIT sequence by means of the Berkowitz, Kolmogorov-Smirnov, and Jarque-Bera distributional tests. Second, we employ the logarithmic scoring rule, which evaluates the accuracy of each method in predicting the ex post realizations. Third, we are the first to apply, to our knowledge, a return-based continuous ranked probability score (CRPS) to financial forecasts. The CRPS compares the realizations to the entire ex ante densities, ranking all methods in terms of their prediction errors. Finally, we develop a new indicator, the integrated forecast score (IFS), which aggregates the results from the statistical consistency, local accuracy, and forecasting errors analyses into a single composite measure.

We calibrate our RNDs using market-derived option prices only. This approach contrasts with the use of exchange-reported settlement prices, which in many cases are theoretically estimated and already reflect specific modeling choices. Finally, we do not consider in this paper combinations of risk-neutral and historical methods; while this approach seems promising, our aim is to shed light on the predictive ability of the most commonly used models in financial economics, thus leaving mixed densities for future research.

CONCLUSIONS

Our paper presents a comprehensive analysis of the most commonly used density schemes in financial economics. Through the development of a novel IFS, we show that RNDs outperform historical-based predictions in terms of information content. The IFS is constructed by aggregating the statistical consistency, local accuracy, and forecasting errors results into a single normalized measure. Using an option dataset covering from 1995 to 2016, we find that the variance gamma model simultaneously delivers the largest out-of-sample log-likelihood and the lowest forecasting errors, thus ranking first in the IFS.

In contrast, the ARCH-based GJR-FHS achieves the best score in statistical consistency, generating the most reliable forecasts across the entire density range. We also find two strong patterns regarding historical models. First, in all density schemes the use of 5-year calibration periods outperforms the forecasting ability of 6-month calibration windows. Second, densities obtained from ARCH-type models are more informative than those generated with lognormal methods or a bootstrapping of historical returns. Conversely, frequently used benchmarks like the Heston model or the nonparametric Breeden-Litzenberger formula yield biased predictions and are rejected in statistical tests.

Looking forward, optimally mixing the information content of risk-neutral and historical schemes, and exploring the use of machine learning algorithms to calibrate such models is worthy of research. Moreover, while the IFS provides a simple solution to a complex verification problem, applying the IFS in other datasets or testing its performance in real trading strategies could help to validatethe usefulness of this measure as a new tool in financial forecasting. These items remain in our agenda for future research.

5.3 Article 3: Crisóstomo, R. (2021). Estimating real-world probabilities: A forward-looking behavioral framework. *Journal of Futures Markets*, 41, 1797-1823. https://doi.org/10.1002/fut.22248

ABSTRACT

We show that disentangling sentiment-induced biases from fundamental expectations significantly improves the accuracy and consistency of probabilistic forecasts. Using data from 1994 to 2017, we analyze 15 stochastic models and risk-preference combinations and in all possible cases a simple behavioral transformation delivers substantial forecast gains. Our results are robust across different evaluation methods, risk-preference hypotheses, and sentiment calibrations, demonstrating that behavioral effects can be effectively used to forecast asset prices. We also implement a trading strategy that shows how behavioral biases can be exploited to generate trading profits. Further analyses confirm that our real-world densities outperform forecasts recalibrated to avoid past mistakes and improve predictive models where risk aversion is dynamically estimated from option prices.

INTRODUCTION

Asset pricing models have evolved under the paradigms of market efficiency and rational expectations. Yet, since the pioneering work of Keynes, increasing evidence shows that investors commit systematicbehavioral errors that manifest through asset prices

We contribute to the literature by developing a forward-looking framework to measure investor sentiment andquantify its effects. Methodologically, we start with the risk-neutral distributions obtained from the most common benchmarks in financial economics, including stochastic volatility models, discontinuous jumps, and nonparametric densities. All risk-neutral predictions are adjusted to incorporate investor's risk preferences through several utility formulations.

We next estimate the sentiment function which summarizes investor biases in specific areas of the return distribution. Following Cochrane and Shefrin, we analyze the impact of behavioral biases through the stochastic discount factor (SDF). The SDF or pricing kernel is the cornerstone of asset pricing, embodying investor preferences and beliefs about future returns. In traditional finance, the SDF must be monotonically decreasing, reflecting a diminishing marginal utility in terms of wealth. However, empirical analyses show that the SDF exhibits a counterintuitive upward-sloping portion, giving rise to the pricing kernel puzzle

When the SDF is expanded to incorporate sentiment effects, the pricing kernel collectively embodies time-discount, risk preferences, and behavioral biases. While the first two are well-known in finance, the behavioral component of the SDF represents the change of measure required to incorporate investor sentiment in different areas of the probabilistic forecast. We consider three sentiment-induced mistakes: Excessive optimism, which generates biases on average returns; overconfidence, which impacts volatility predictions, and tail sentiment, which is related to nonrational tail expectations.

We calibrate our sentiment function using simple market-based inputs. Investor optimism is derived from changes in implied volatilities; overconfidence is proxied by changes in trading volumes, and tail sentiment is obtained from the skewness of the risk-neutral distribution. All investor biases are then aggregated into an ex ante sentiment function that is used to transform risk-adjusted forecasts into real-world densities. To gauge the impact of sentiment effects, weconsider two alternative calibrations: a low impact and a high impact specification.

We then examine the out-of-sample performance of all density forecasts. The accuracy of each model is assessed through the log-likelihood score; forecast errors are evaluated in terms of the continuous ranked probability score (CRPS), and statistical consistency is measured with the Berkowitz, Jarque‑Bera (JB), and Kolmogorov‑Smirnov (KS) tests. Finally, we summarize all forecast metrics in a single ranking using the integrated forecast score (IFS).

Our results are striking. We analyze 15 stochastic models and risk-preference combinations and in all possible cases a simple behavioral correction generates substantial forecast gains. Remarkably, the improvement delivered by our real-world transformation is robust across all evaluation methods, risk-preference hypotheses, and sentiment calibrations, demonstrating that sentiment effects can be effectively used to forecast future prices. Furthermore, we implement a trading strategy which shows how behavioral trading can be used to generate substantial trading profits and excess returns.

We also perform two additional tests. First, we show that our real-world densities outperform nonparametric forecasts that have been recalibrated to avoid past mistakes. Second, we show that behavioral corrections also improve the explanatory power of density predictions where risk aversion is dynamically estimated from option prices.

CONCLUSIONS

This paper examines whether investor sentiment can be used to improve the forecasting ability of density predictions obtained from option prices. Increasing evidence shows that real-world investors commit systematic behavioral errors that manifest in asset prices. Consequently, it follows that market-implied forecasts should be appropriately corrected to disentangle the impact of behavioral biases from fundamental expectations.

To quantify sentiment effects, we develop a forward-looking framework that generates the behavioral correction required to adjust traditional forecasts in specific areas of the return distribution. For 15 underlying models and riskpreference combinations, we show that a simple behavioral transformation in the mean, variance, and tail estimates of traditional predictions significantly improve their accuracy and statistical consistency.

Information gains are robust across all forecast metrics and sentiment calibrations, demonstrating that behavioral effects can be effectively used to predict asset prices. To quantify the benefits of behavioral trading, we implement a trading strategy that achieves substantial trading profits and excess returns. Our results also show that real-world densities outperform non-parametric corrections derived

6 Conclusions

This thesis studies the mathematical methods used to forecast future asset prices.

We first consider the numerical challenges that complicate the estimation of density predictions through Fourier-based methods. Crisóstomo (2018) shows that different Fourier methods exhibit notably different biases. We find that among quadrature-based methods: (i) the CM-OPT stands out for minimizing truncation errors in the Fourier space; (ii) the DPD-OPT exhibits the highest sampling efficiency but also the slowest decay rate; and (iii) the AT-OPT suffer the largest discretization errors, requiring higher values of N to achieve comparable accuracy levels. Overall, the DPD-OPT performs best when pricing a high number of options, the CM-OPT is more efficient when only a few prices are required, while the AT-OPT typically ranks as the slowest strike-optimized alternative.

Compared to traditional Fourier methods, our analyses demonstrate the efficiency of strike vector computations. In our tests, computing option prices through the AT-OPT, CM-OPT, DPD-OPT, and COS-OPT is, on average, 54, 67, 165, and over 1500 times faster than in the FFT. Across all methods, the multi-strike version of the COS formula is the overall fastest alternative, a result that stems from the lower truncation range required in the COS method and rapid decay of the cosine series coefficients.

In terms of new pricing algorithms, in Crisóstomo (2018) we are the first to propose and evaluate a strike-optimized version of the Carr-Madan formula, showing that the CM-OPT significantly improves the speed and accuracy of the FFT. While both are based on the same pricing approach, the CM-OPT's flexibility allows: (i) pricing any required strikes; (ii) choosing any integration grid and technique; and (iii) avoiding interpolation biases. As a result, the CM-OPT is both faster and more accurate than the FFT, thus rendering this method inefficient.

We also show that obtaining accurate option values can be particularly challenging in the VG model. While all methods converge under jump-diffusion dynamics, large truncation errors significantly complicate the practical VG implementation. Moreover, depending on the VG parameters, specific Fourier implementations may completely fail to provide reasonable option prices.

Next, we perform a comprehensive analysis of the most commonly used density models to forecast asset prices. Through a unique dataset of option prices covering from 1995 to 2016, Crisóstomo and Couso (2018) show that risk-neutral methods outperform historical-based densities in terms of information content. Among the specific choice, we find that the Variance Gamma model delivers the largest out-of-sample log-likelihood and the lowest forecasting errors. In contrast, the ARCH-based GJR-FHS achieves the best score in statistical consistency, generating the most reliable forecasts across the entire density range.

Contributing to the probabilistic forecast literature, Crisóstomo and Couso (2018) develops a novel Integrated Forecasting Score (IFS) which aggregates in a standardized [0-1] scale the results from statistical consistency, local accuracy, and forecasting errors analyses. Using the IFS, we find two strong patterns regarding historical models. First, in all density schemes, the use of 5-year calibration periods outperforms the forecasting ability of 6-month calibration windows. Second, densities obtained from ARCH-type models are more informative than those generated with lognormal methods or a bootstrapping of historical returns. Conversely, frequently used risk-neutral benchmarks like the Heston model or the non-parametric Breeden-Litzenberger formula yield biased predictions and are rejected in statistical tests.

Once historical and risk-neutral models have been assessed, we examine whether investor risk preferences and investor sentiment can be used to improve the forecasting ability of density predictions obtained from option prices. In terms of investors' preferences, our results show that including a moderate amount of risk aversion improves the accuracy and statistical consistency of probabilistic forecast, a finding that is in line with the literature.

In contrast, there are much fewer analyses that consider how to incorporate sentiment-induced biases in density predictions. Given the increasing evidence showing that real-world investors commit systematic behavioral errors that manifest in asset prices, market-implied forecasts should be appropriately corrected to disentangle behavioral biases from fundamental expectations.

In Crisóstomo (2021), we propose a novel forward-looking framework to quantify the impact of sentiment in different areas of the return distribution. We develop a time-variant adjustment that considers three sentiment-induced mistakes: Excessive optimism, which generates biases on average returns; overconfidence, which impacts volatility predictions, and tail sentiment, which is related to non-rational tail expectations-

For a wide range of underlying models and risk-preference combinations, we show that disentangling sentiment-induced biases from fundamental expectations significantly improves the accuracy and consistency of probabilistic forecasts. Specifically, a simple behavioral transformation in the mean, variance, and tail estimates of traditional predictions delivers information gains across all forecast metrics and sentiment calibrations, demonstrating that behavioral effects can be effectively used to predict asset prices.

To complement the forecast gains in statistical terms, we quantify the practical benefits of our sentiment framework with a market trading strategy. In particular, we implement a behavioral strategy designed to exploit the misspecification of traditional densities by going long states with too-low probability and short states with too-high probability. Through option positions that are initiated when excessive optimism, confidence, or tail biases are embedded in market prices, we show that behavioral trading can achieve notable profits and excess returns. Overall, the return of our sentiment-based strategy is almost twice the expected return under the utility-adjusted CRRA benchmark, and 28 times higher than the return of a risk-free investment.

We also perform two additional robustness tests. First, we compare the forecasting ability of our behavioral adjustment with a standard method to correct density predictions in light of past mistakes. Our analyses show that forward-looking sentiment adjustments outperform statistical recalibrations in terms of information content. Second, we explore the performance of our behavioral transformation when applied to traditional densities where an implied risk aversion is estimated from option prices, demonstrating that behavioral adjustments improve neoclassical densities with a time-varying risk aversion.

Regarding future research, the results of our sentiment-based framework could be extended to other markets. While the focus of our thesis is the Spanish equity benchmark, the IBEX 35 Index, our sentiment framework could be easily replicable in other markets with liquid financial derivatives. For instance, it would be interesting to see forecast improvements and trading gains in the EURO STOXX 50 index or the S&P 500. Similarly, the improved results as we move from historical to risk-neutral, risk-adjusted, and behavioral-adjusted densities, while theoretically consistent, could be contrasted in other markets.

In terms of modelling techniques, it is also worth exploring whether mixing the information from historical and option-implied markets can improve the forecasting power of density predictions for asset prices. Although we follow a compartmental approach to model categories, the development of new calibration and forecasting algorithms based on machine learning can provide alternative ways to optimally mix the inputs from different markets and model categories. The potential gains of such mixed models should be weighed, however, against the potential loss of explainability and model transparency.

Next, in terms of the evaluation measures, the IFS provides a simple and easily understandable framework to aggregate partial evaluation metrics and assess the global accuracy of a forecasting scheme. However, further analyses could explore the interrelation between the partial measures. For example, while measuring different aspects of forecast accuracy, the log sore and the CRPS are typically correlated; hence further optimizing the IFS weights could potentially improve its discrimination power. Similarly, instead of using fixed α significance levels for the goodness-of-fit tests, an approach based on the α-values could avoid jumps in the ranking measures, as two models with marginally better/worse p-values should not be ranked too distantly.

In terms of Fourier-based algorithms, the higher CM-OPT efficiency compared with the FFT should be expected in any underlying model by construction. In contrast, the improvement delivered by the COS method could be tested in the stochastic processes that have been gaining traction in recent years. In addition, it would be interesting to analyze the pricing biases and computational efficiency of the new generation of rough volatility models[4], which promise modeling notable advantages over traditional stochastic models (e.g.: being able to appropriately represent the long-dated dynamics of the volatility smile).

[444] See, Gatheral, Jaisson, and Rosenbaum (2018) and Bayer, Friz, and Gatheral (2016).

Finally, since the comparison in Crisóstomo (2018), new pricing schemes which rely on local wavelet bases, like the WA of Ortiz-Gracia and Oosterlee (2013) and the PROJ method of Kirkby (2015) (both based on B-spline wavelets) or the SWIFT[5] method of Ortiz-Gracia and Oosterlee (2016) (which employs Shannon Wavelets), have emerged as efficient pricing alternatives. These methods, however, require a more involved computation which leads to specific implementation challenges. Consequently, it would also be interesting to benchmark how these models perform compared to quadrature schemes or the COS method, and analyze the pricing biases that arise when local wavelet bases are employed to evaluate options under different stochastic price processes.

All these topics remain in our agenda for further research.

[55] Shannon Wavelets Inverse Fourier Technique.

7 Abstract

This thesis deals with the mathematical methods used to forecast future asset prices. First, we analyze the numerical challenges that complicate obtaining density predictions through Fourier-based methods. We find that different Fourier implementations lead to notable different truncation, discretization, and interpolation biases. Among the different methods, we find that the COS method is the fastest overall alternative. Furthermore, we develop a strike-optimized version of the Car-Madan formula, which is simultaneously faster and more accurate than the FFT, thus rendering this method inefficient.

Next, we explore the forecasting power of the most common used benchmark in quantitative finance. Overall, we find that risk neutral methods outperform historical-based predictions in terms of information content, with the Variance Gamma being the most accurate stochastic process across a range of forecast criteria. Since current metrics to assess probabilistic predictions analyze partial aspects of forecast performance, we develop a comprehensive methodology, called Integrated Forecast Score, which aggregates in a standardized [0,1] scale the local accuracy, global errors, and statistical consistency of different predictive schemes.

We also analyze how density predictions obtained from option prices can be improved by incorporating investors' risk preferences and sentiment effects. In terms of risk preferences, we find a clear improvement when a moderate amount of risk aversion is considered. Regarding sentiment, we develop a novel method based on the stochastic discount factor framework to quantify the impact of investor biases in different areas of the return distribution. Specifically, we consider three sentiment-induced biases: excessive optimism, which generates biases on average returns; overconfidence, which impacts volatility predictions, and tail sentiment, which is related to non-rational tail expectations. Through a simple behavioral transformation in the mean, variance, and tail estimates of traditional predictions, we show that disentangling sentiment-induced biases from fundamental expectations delivers information gains across all forecast metrics and sentiment calibrations.

Finally, we develop a trading strategy that exploits the misspecification of traditional densities by going long states with too-low probability and short states with too-high probability. Using option-based positions that are initiated when excessive optimism, confidence, or tail biases are embedded in market prices, we show that the return of behavioral trading is almost twice the expected return under the utility-adjusted CRRA benchmark, and 28 times higher than the return of a risk-free investment.

Resumen

Esta tesis trata sobre los métodos matemáticos utilizados para estimar el precio futuro de los activos financieros. En primer lugar, se analizan los desafíos numéricos que complican la obtención de predicciones de funciones de densidad a través de métodos basados en transformadas de Fourier. Nuestros análisis muestran que diferentes implementaciones basadas en transformadas de Fourier generan sesos de truncamiento, discretización e interpolación sustancialmente diferentes. A nivel global, el método COS resulta el más rápido computacionalmente de todos los algoritmos. Además, se prone una versión con vectorización de strikes que la fórmula de Carr-Madan que es simultáneamente más rápida y precisa que la transformada rápida de Fourier (FFT), haciendo que este método no resulte eficiente.

A continuación se contrasta el poder predictivo de los modelos de evolución de activos más utilizado en finanzas cuantitativas. En general, encontramos que las modelos riesgo-neutro superan a las estimaciones basadas en datos históricos en términos de contenido de información, siendo el método *Variance Gamma* el proceso estocástico más predictivo conforme a diferentes criterios de medición. Puesto que actualmente las métricas para medir la capacidad predictiva de funciones de densidad actualmente analizan aspectos parciales de cada estimación, se propone una metodología global, denominada *Integrated Forecast Score*, que agrega en una única escala estandarizada la precisión local, los errores globales y la consistencia estadística de los métodos predictivos.

Posteriormente se analiza cómo mejorar las predicciones de densidad obtenidas a partir de precios de opciones incorporando las preferencias de riesgo de los inversores y los efectos del sentimiento. En términos de preferencias de riesgo, encontramos que la estimación mejora de forma notable cuando se considera una aversión al riesgo moderada. En cuanto al sentimiento, se desarrolla un nuevo método basado en el factor de descuento estocástico para cuantificar el impacto de los sesgos conductuales de los inversores en diferentes áreas de la distribución de la rentabilidad. En particular, nuestro análisis considera tres sesgos inducidos por el sentimiento: Optimismo excesivo, que genera sesgos en el rendimiento medio esperado; exceso de confianza, que afecta las predicciones de volatilidad, y sentimiento de cola, que está relacionado con expectativas no racionales en las colas de la distribución. Mediante una transformación de en la media, la varianza y las colas de la distribución, nuestro análisis muestra que corregir los sesgos de sentimiento mejora consistentemente la capacidad predictiva de las estimaciones realizadas con modelos de racionalidad perfecta.

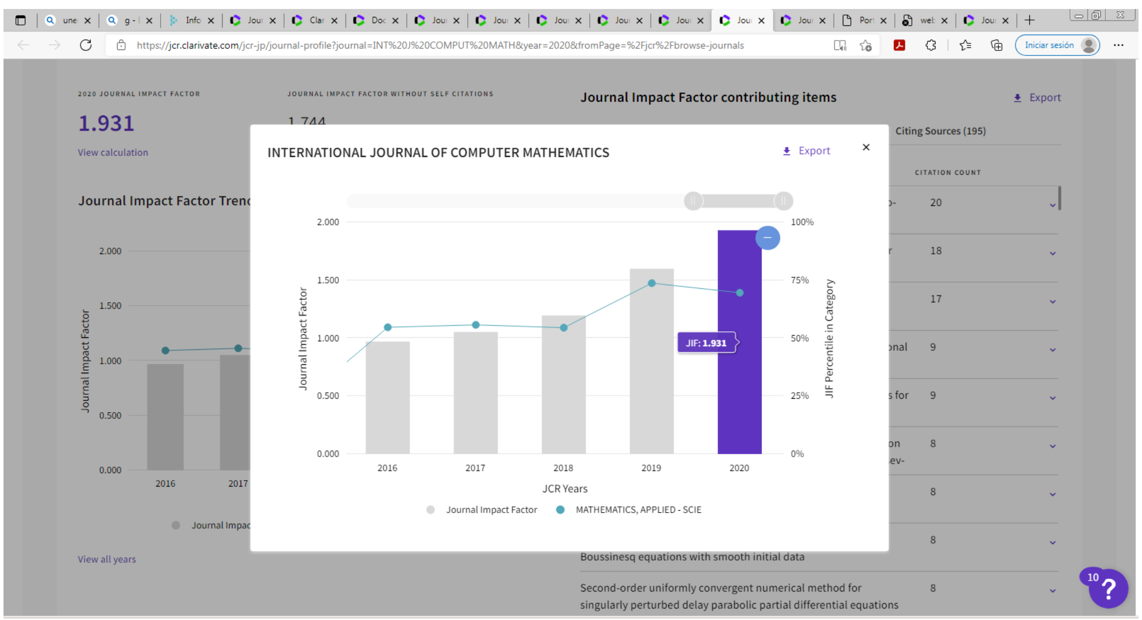
Finalmente, desarrollamos una estrategia de negociación que explota los sesgos de las funciones de densidad tradicionales tomando posiciones largas en las áreas de la distribución con una probabilidad excesivamente baja y posiciones cortas en las áreas con una probabilidad excesivamente alta. Mediante posiciones en opciones que se inician cuando el mercado muestra un exceso de optimismo, confianza o sesgos de cola, encontramos que el rendimiento de nuestra

estrategia en función del sentimiento es casi dos veces el retorno esperado conforme a la función de utilidad más predictiva y 28 veces mayor que el rendimiento de una inversión libre de riesgo.

8 Impact factor and quartile of the academic journals

This Chapter contains the Journal Impact Factor (JIF), the JIF percentile and the associated quartile of the journals in which the publications derived from this thesis have been published. These figures are obtained directly from Web of Science (WoS), which is the official provider of the Journal Citation Reports (JCR).

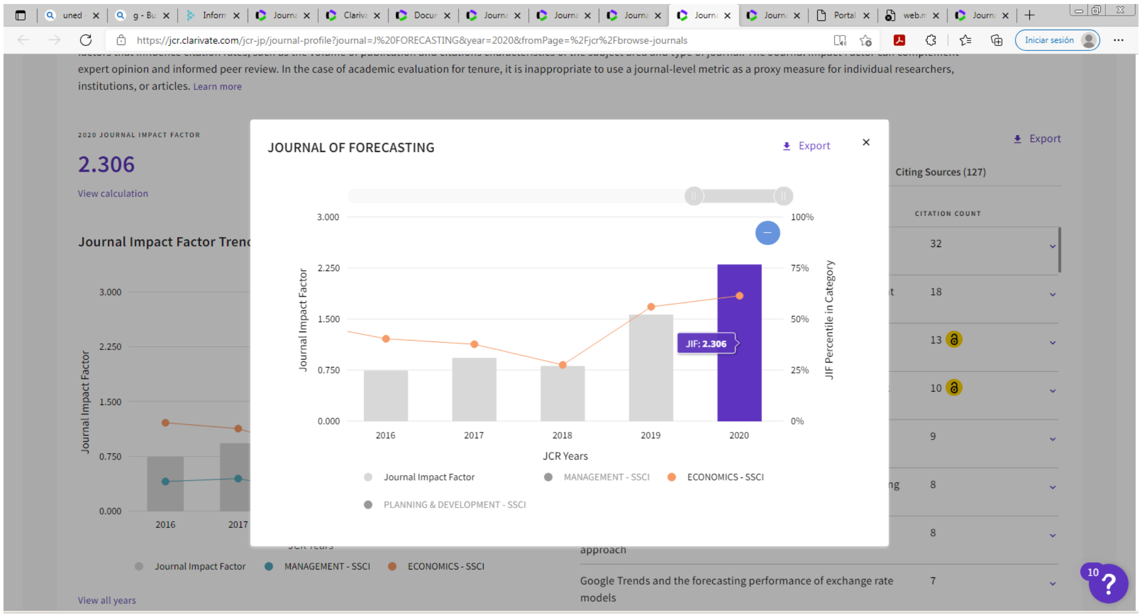
Article 1: Crisóstomo, R. (2018): Speed and biases of Fourier-based pricing choices: a numerical analysis. *International Journal of Computer Mathematics*, 95:8, 1565-1582. https://doi.org/10.1080/00207160.2017.1322691



The *International Journal of Computer Mathematics* has a 2020 JCR impact factor of 1.931, and it is located in the 70th JIF percentile, which corresponds to the second quartile (Q2) of the JCR category Applied Mathematics (SCIE).

The impact factor of the *International Journal of Computer Mathematics* has increased steadily since the publications of Crisóstomo, R. (2018), starting at 1.196 in 2018 and reaching 1.931 in the last JCR report. This article has received 9 citations (Google Scholar) up to March 2022.

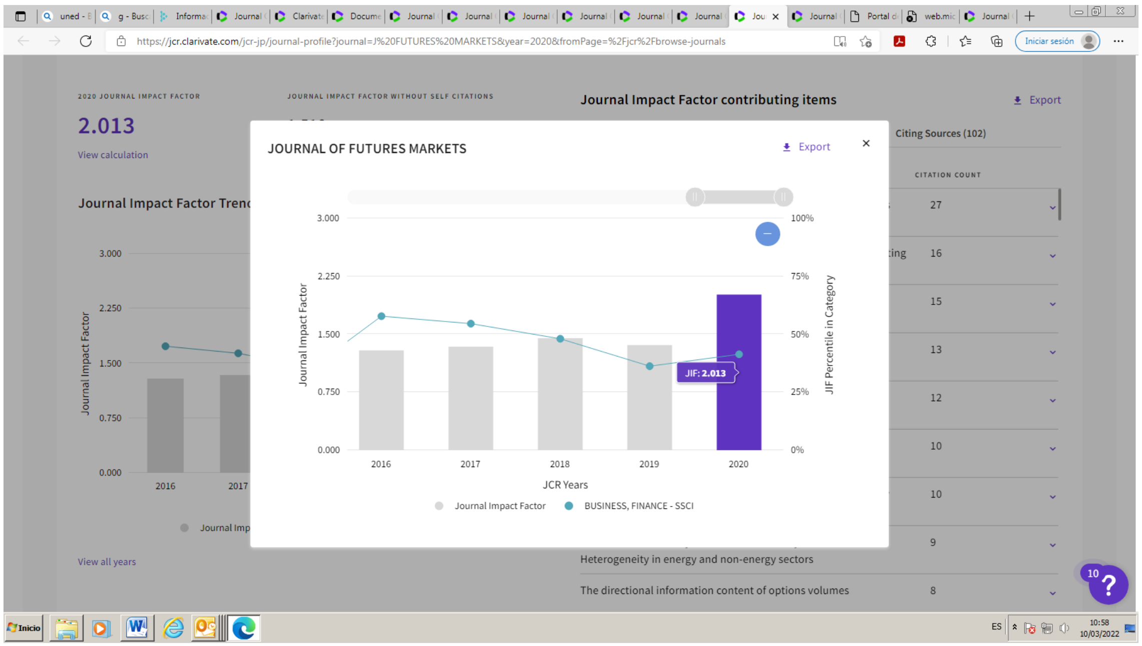
Article 2: Crisóstomo, R. and Couso, L. (2018), Financial density forecasts: A comprehensive comparison of risk-neutral and historical schemes. *Journal of Forecasting* 37: 589-603. https://doi.org/10.1002/for.2521



The *Journal of Forecasting* has a 2020 JCR impact factor of 2.306, and it is located in the 62th JIF percentile, which corresponds to the second quartile (Q2) of the JCR category Economics (SSCI).

The impact factor of the *Journal of Forecasting* has increased steadily since the publications of Crisóstomo, R. and Couso, L. (2018), starting at 0.816 in 2018 and reaching 2.306 in the last JCR report. This article has received 4 citations (Google Scholar) up to March 2022.

Article 3: Crisóstomo, R. (2021). Estimating real-world probabilities: A forward-looking behavioral framework. *Journal of Futures Markets*, 41, 1797– 1823. https://doi.org/10.1002/fut.22248



The *Journal of Futures Markets* has a 2020 JCR impact factor of 2.013, and it is situated in the 41th JIF percentile, which corresponds to the second quartile (Q3) of the JCR category Business, Finance (SSCI).

The impact factor of the *Journal of Futures Markets* has increased in the last years, rising from 1.291 in 2016 to 2.013 in the last JCR report. This article was published in July 2021 and has received 1 citation (Semantic Scholar) up to March 2022.

9 Other scientific contributions derived from this thesis

In addition to the JCR publications included in Chapter 5, the article entitled “An Analysis of the Heston Stochastic Volatility Model: Implementation and Calibration using Matlab” was published during the formative period of this thesis. This article analyzes the implementation of the Heston model, tests different calibration routines, and provides an analytic description of how characteristic functions can be used to obtain option prices. The scientific contribution of this paper has been widely recognized in the academic community, and this article has received 38 citations (Google Scholar) up to March 2022.

The full reference of this article is the following:

- Crisóstomo, R (2014): An Analysis of the Heston Stochastic Volatility Model: Implementation and Calibration using Matlab. *CNMV Working Paper* 58: 1-46. http://dx.doi.org/10.2139/ssrn.2527818

10 Bibliography

Abreu, Margarida, and Victor Mendes, 2012, Information, overconfidence and trading: Do the sources of information matter?, Journal of Economic Psychology 33, 868–881.

Aït-Sahalia, Yacine, and Andrew W. Lo, 2000, Nonparametric risk management and implied risk aversion, Journal of Econometrics 94, 9–51.

Albrecher, Hansjörg, Philipp Mayer, Wim Schoutens, and Jurgen Tistaert, 2007, The little Heston trap, Wilmott Magazine, 83–92.

Alonso, Francisco, Roberto Blanco, and Gonzalo Rubio, 2005, Testing the Forecasting Performance of Ibex 35 Option-Implied Risk-Neutral Densities, Bank of Spain Working Papers 0504.

Alonso, Francisco, Roberto Blanco, and Gonzalo Rubio, 2009, Option-implied preferences adjustments, density forecasts, and the equity risk premium, Spanish Economic Review 11, 141–164.

Amisano, G, and R Giacomini, 2007, Comparing density forecasts via weighted likelihood ratio tests, Journal of Business & Economic Statistics 25, 177–190.

Anagnou, I, M. Bedendo, Stewart Hodges, and Robert G. Tompkins, 2005, Forecasting accuracy of implied and GARCH-based probability density functions, Review of Futures Markets 11, 41–66.

Anagnou, Iliana, Mascia Bedendo, Stewart Hodges, and Robert Tompkins, 2003, The Relation Between Implied and Realised Probability Density Functions, Financial Options Research Centre, University of Warwick.

Attari, Mukarram, 2004, Option Pricing Using Fourier Transforms : A Numerically Efficient Simplification, SSRN Electronic Journal.

Baker, Malcolm, and Jeffrey Wurgler, 2006, Investor sentiment and the cross-section of stock returns, Journal of Finance 61, 1645–1680.

Baker, Malcolm, and Jeffrey Wurgler, 2007, Investor sentiment in the stock market, Journal of Economic Perspectives 21.

Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 1997, Empirical Performance of Alternative Option Pricing Models, The Journal of Finance 52, 2003–2049.

Bakshi, Gurdip, Nikunj Kapadia, and Dilip Madan, 2003, Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options, Review of Financial Studies 16, 101–143.

Bakshi, Gurdip, and Dilip Madan, 2006, A theory of volatility spreads, Management Science 52, 1945–1956.

Bakshi, Gurdip, and Dilip B. Madan, 2000, Spanning and derivative-security valuation, Journal of Financial Economics 55, 205–238.

Bao, Yong, Tae Hwy Lee, and Burak Saltoğlu, 2007, Comparing density forecast models, Journal of Forecasting 26, 203–225.

Barber, Brad M., and Terrance Odean, 2001, Boys will be boys: Gender, overconfidence, and common stock investment, Quarterly Journal of Economics 116, 261–292.

Barone-Adesi, Giovanni, Robert F. Engle, and Loriano Mancini, 2008, A GARCH option pricing model with filtered historical simulation, Review of Financial Studies 21, 1223–1258.

Barone-Adesi, Giovanni, Chiara Legnazzi, and Carlo Sala, 2018, S&P 500 Index, an Option-Implied Risk Analysis, Swiss Finance Institute Research Paper Series N°18-29.

Barone-Adesi, Giovanni, Loriano Mancini, and Hersh Shefrin, 2017, "Estimating Sentiment, Risk Aversion, and Time Preference from Behavioral Pricing Kernel Theory": Online Appendix, SSRN Electronic Journal.

Bates, David S., 1996, Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options, Review of Financial Studies 9, 69–107.

Bayer, Christian, Peter Friz, and Jim Gatheral, 2016, Pricing under rough volatility, Quantitative Finance 16.

Beare, Brendan K., 2011, Measure preserving derivatives and the pricing kernel puzzle, Journal of Mathematical Economics 47, 689–697.

Beare, Brendan K., and Lawrence D.W. Schmidt, 2016, An Empirical Test of Pricing Kernel Monotonicity, Journal of Applied Econometrics 31, 338–356.

Berkowitz, Jeremy, 2001, Testing Density Forecasts, With Applications to Risk Management, Journal of Business & Economic Statistics 19, 465–474.

Bevilacqua, Mattia, Iftekhar Hasan, and Radu Tunaru, 2018, The SKEW Index: Extracting What Has Been Left, SSRN Electronic Journal.

Black, Fischer, and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, Journal of Political Economy 81, 637.

Blair, Bevan J., Ser Huang Poon, and Stephen J. Taylor, 2001, Forecasting S&P 100 volatility: The incremental information content of implied volatilities and high-frequency index returns, Journal of Econometrics 105, 5–26.

Bliss, Robert R., and Nikolaus Panigirtzoglou, 2004, Option-Implied Risk Aversion Estimates, Journal of Finance 59, 407–446.

Bollen, Nicolas P.B., and Robert E. Whaley, 2004, Does Net Buying Pressure Affect the Shape of Implied Volatility Functions?, Journal of Finance 59, 711–753.

Bollerslev, Tim, and Viktor Todorov, 2011, Tails, Fears, and Risk Premia, Journal of Finance 66, 2165–2211.

Bollerslev, Tim, and Jeffrey M. Wooldridge, 1992, Quasi-maximum likelihood estimation and inference in dynamic models with time-varying covariances, Econometric Reviews 11, 143–172.

Bondarenko, Oleg, 2014, Why Are Put Options so Expensive?, Quarterly Journal of Finance 4.

Breeden, Douglas T., and Robert H. Litzenberger, 1978, Prices of State-Contingent Claims Implicit in Option Prices, The Journal of Business 51, 621–651.

Brinkmann, Felix, and Olaf Korn, 2018, Risk-adjusted option-implied moments, Review of Derivatives Research 21, 149–173.

Broadie, Mark, and Ozgur Kaya, 2006, Exact Simulation of Stochastic Volatility and Other Affine Jump Diffusion Processes, Operations Research 54, 217–231.

Brown, David P., and Jens Carsten Jackwerth, 2012, The pricing kernel puzzle: Reconciling index option data and economic theory, Contemporary Studies in Economic and Financial Analysis 94, 155–183.

Busch, Thomas, Bent Jesper Christensen, and M. Ø Nielsen, 2011, The role of implied volatility in forecasting future realized volatility and jumps in foreign exchange, stock, and bond markets, Journal of Econometrics 160, 48–57.

Canina, Linda, and Stephen Figlewski, 1993, The informational content of implied volatility, The Review of Financial Studies 6, 659–681.

Carr, Peter, and Dilip B. Madan, 1999, Option valuation using the fast Fourier transform, Journal of Computational Finance 3, 463–520.

Carr, Peter, and Liuren Wu, 2003, The finite moment logstable process and option pricing, The Journal of Finance 58, 753–778.

Chen, Bei, and Quan Gan, 2018, Sentiment, Implied Volatility Slope, and Risk-Neutral Skewness, SSRN Electronic Journal.

Chourdakis, Kyriakos, 2005, Option Pricing Using The Fractional FFT, Journal of Computational Finance 7, 1–23.

Christoffersen, Peter., 2012, Elements of Financial Risk Management, Second Edition (Elsevier).

Christoffersen, Peter, Kris Jacobs, and Bo-Young Chang, 2013, Forecasting with option-implied information, Handbook of Economic Forecasting 2, 581–656.

Cochrane, John H., 2005, Asset Pricing (Revised Edition) Princeton University Press.

Constantinides, George M., Jens Carsten Jackwerth, and Stylianos Perrakis, 2009, Mispricing of S&P 500 index options, Review of Financial Studies 22, 1247–1277.

Cont, Rama, 2001, Empirical properties of asset returns: stylized facts and statistical issues, Quantitative Finance 1, 223–236.

Cont, Rama, and Peter Tankov, 2004, Financial Modelling with Jump Processes (Chapman & Hall/CRC).

Crisóstomo, Ricardo, 2014, An Analysis of the Heston Stochastic Volatility Model: Implementation and Calibration using Matlab, CNMV Working papers No. 58.

Crisóstomo, Ricardo, 2018, Speed and biases of Fourier-based pricing choices: a numerical analysis, International Journal of Computer Mathematics 95, 1565–1582.

Crisóstomo, Ricardo, 2021, Estimating real-world probabilities: A forward-looking behavioral framework, Journal of Futures Markets 41, 1797–1823.

Crisóstomo, Ricardo, and Lorena Couso, 2018, Financial density forecasts: A comprehensive comparison of risk-neutral and historical schemes, Journal of Forecasting 37, 589–603.

Cuesdeanu, Horatio, and Jens Carsten Jackwerth, 2018, The pricing kernel puzzle: survey and outlook, Annals of Finance 14, 289–329.

Cuesdeanu, Horatio, and Jens Carsten Jackwerth, 2018, The pricing kernel puzzle in forward looking data, Review of Derivatives Research 21, 253–276.

Da, Zhi, Joseph Engelberg, and Pengjie Gao, 2015, The sum of all FEARS investor sentiment and asset prices, Review of Financial Studies.

De Bondt, Werner F.M., and Richard H. Thaler, 1995, Financial Decision-Making in Markets and Firms: A Behavioral Perspective, Handbooks in Operations Research and Management Science.

De Bondt, Werner, Yaz Muradoglu, Hersh Shefrin, and Sotiris Staikouras, 2015, Behavioral Finance: Quo Vadis?, Journal of Applied Finance 18, 1–15.

De Long, J. Bradford, Andrei Shleifer, Lawrence H. Summers, and Robert J. Waldmann, 1990, Noise Trader Risk in Financial Markets, Journal of Political Economy 98, 703–738.

de Vincent-Humphreys, Rupert, and Joeseph Noss, 2012, Estimating probability distributions of future asset prices: empirical transformations from option-implied risk-neutral to real-world density functions, Bank of England Working Papers 455.

DeMiguel, Victor, Yuliya Plyakha, Raman Uppal, and Grigory Vilkov, 2013, Improving portfolio selection using option-implied volatility and skewness, Journal of Financial and Quantitative Analysis 48, 1813–1845.

Diebold, Francis X., Todd Gunther, and Anthony Tay, 1998, Evaluating density forecasts with applications to financial risk management, International Economic Review 39, 863–883.

Dowd, Kevin, 2004, A modified Berkwitz back-test, Risk magazine 17, 86–87.

Duffie, Darrell, Jun Pan, and Kenneth J Singleton, 2000, Transform Analysis and Asset Pricing for Affine Jump-Diffusions, Econometrica 68, 1343–1376.

Fackler, Paul L., and Robert P. King, 1990, Calibration of Option-Based Probability Assessments in Agricultural Commodity Markets, American Journal of Agricultural Economics.

Fang, F., and C. W. Oosterlee, 2009, A Novel Pricing Method for European Options Based on Fourier-Cosine Series Expansions, SIAM Journal on Scientific Computing 31, 826–848.

Figlewski, Stephen, 2018, Risk-Neutral Densities: A Review, Annual Review of Financial Economics 10, 329–359.

Fiorani, Filo, 2004, Option Pricing Under the Variance Gamma Process, MPRA Paper No. 15395.

Gatheral, Jim, Thibault Jaisson, and Mathieu Rosenbaum, 2018, Volatility is rough, Quantitative Finance 18.

Giot, Pierre, 2005, Relationships between implied volatility indexes and stock index returns, Journal of Portfolio Management 31.

Giot, Pierre, and Sébastien Laurent, 2007, Information content of implied volatility in light of the jump / continuous decomposition of realized volatility, The Journal of Futures Markets 27, 337–359.

Glaser, Markus, and Martin Weber, 2007, Overconfidence and trading volume, GENEVA Risk and Insurance Review 32, 1–36.

Glosten, Lawrence R., Ravi Jagannathan, and David E. Runkle, 1993, On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, Journal of Finance 48, 1779–1801.

Gneiting, Tilmann, and Matthias Katzfuss, 2014, Probabilistic Forecasting, Annual Review of Statistics and Its Application 1, 125–151.

Gneiting, Tilmann, and Adrian E Raftery, 2007, Strictly Proper Scoring Rules, Prediction, and Estimation, Journal of the American Statistical Association 102, 359–378.

Göncü, Ahmet, Mehmet O. Karahan, and Tolga U. Kuzubaş, 2016, A comparative goodness-of-fit analysis of distributions of some Lévy processes and Heston model to stock index returns, North American Journal of Economics and Finance 36, 69–83.

Greenwood-Nimmo, Matthew, Viet Hoang Nguyen, and Barry Rafferty, 2016, Risk and return spillovers among the G10 currencies, Journal of Financial Markets 31, 43–62.

Grinblatt, Mark, and Matti Keloharju, 2009, Sensation seeking, overconfidence, and trading activity, Journal of Finance.

Grith, Maria, Wolfgang K. Härdle, and Volker Krätschmer, 2017, Reference-dependent preferences and the empirical pricing kernel puzzle, Review of Finance 21, 269–298.

Han, Bing, 2008, Investor sentiment and option prices, Review of Financial Studies 21, 387–414.

Hansen, Peter R., and Asger Lunde, 2005, A forecast comparison of volatility models: Does anything beat a GARCH(1,1)?, Journal of Applied Econometrics 20, 873–889.

Hersbach, Hans, 2000, Decomposition of the Continuous Ranked Probability Score for Ensemble Prediction Systems, Weather and Forecasting 15, 559–570.

Heston, Steven, 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, The Review of Financial Studies 6, 327–343.

Hirsa, Ali, 2012, Computational Methods in Finance (Chapman & Hall/CRC).

Høg, Esben, and Leonidas Tsiaras, 2011, Density forecasts of crude-oil prices using option-implied and ARCH-type models, Journal of Futures Markets 31, 727–754.

Itkin, Andrey, 2010, Pricing options with VG model using FFT, arXiv working paper.

Ivanova, Vesela, and Josep Maria Puigvert Gutiérrez, 2014, Interest rate forecasts, state price densities and risk premium from Euribor options, Journal of Banking and Finance 48, 210–223.

Jackwerth, Jens Carsten, 2000, Recovering Probabilities and Risk Aversion from Options Prices and Realized Returns, The Review of Financial Studies 13, 1–34.

Jiang, George J., and Yisong S. Tian, 2005, The model-free implied volatility and its information content, Review of Financial Studies 18, 1305–1342.

Jones, Christopher S., 2003, The dynamics of stochastic volatility: Evidence from underlying and options markets, Journal of Econometrics 116, 181–224.

Kang, Byung Jin, and Tong Suk Kim, 2006, Option-implied risk preferences: An extension to wider classes of utility functions, Journal of Financial Markets 9, 180–198.

Kang, Byung Jin, Tong Suk Kim, and Sun Joong Yoon, 2010, Information content of volatility spreads, Journal of Futures Markets 30, 533–558.

Kilin, Fiodar, 2011, Accelerating the Calibration of Stochastic Volatility Models, The Journal of Derivatives 18, 7–16.

Kirkby, Justin Lars, 2015, Efficient option pricing by frame duality with the fast Fourier transform, SIAM Journal on Financial Mathematics 6, 713–747.

Kostakis, Alexandros, Nikolaos Panigirtzoglou, and George S. Skiadopoulos, 2011, Market Timing with Option-Implied Distributions: A Forward-Looking Approach, Management Science 57, 1231–1249.

Kullback, S., and R. A. Leibler, 1951, On Information and Sufficiency, The Annals of Mathematical Statistics 22, 79–86.

Lamb, Heather K., David K. Stammers, and Alastair R. Hawkins, 2008, Dinucleotide-sensing proteins: Linking signaling networks and regulating transcription, Science Signaling.

Lee, Roger W., 2004, Option Pricing by Transform Methods: Extensions, Unification, and Error Control, Journal of Computational Finance 7, 51–86.

Lewis, Alan, 2001, A simple option formula for general jump-diffusion and other exponential Lévy processes, OptionCity.net working paper.

Lewis, Alan, 2013, Variance Gamma - different results, Wilmott Forums.

Lin, Chu Bin, Robin K Chou, and George H.K. Wang, 2018, Investor sentiment and price discovery: Evidence from the pricing dynamics between the futures and spot markets, Journal of Banking and Finance 90, 17–31.

Linn, Matthew, Sophie Shive, and Tyler Shumway, 2018, Pricing kernel monotonicity and conditional information, Review of Financial Studies 31, 491–531.

Liu, Xiaoquan, Mark B. Shackleton, Stephen J. Taylor, and Xinzhong Xu, 2007, Closed-form transformations from risk-neutral to real-world distributions, Journal of Banking and Finance 31, 1501–1520.

Lord, Roger, and Christian Kahl, 2010, Complex logarithms in Heston-like models, Mathematical Finance 20, 671–694.

Madan, Dilip B., Peter P. Carr, and Eric C. Chang, 1998, The Variance Gamma Process and Option Pricing, Review of Finance 2, 79–105.

Madan, Dilip B., and Eugene Seneta, 1990, The Variance Gamma (V.G.) Model for Share Market Returns, The Journal of Business 63, 511–524.

Malz, Allan M, 2014, A Simple and Reliable Way to Compute Option-Based Risk-Neutral Distributions, Federal Reserve Bank of New York Staff Reports 677.

Matheson, J. E., and R. L. Winkler, 1976, Scoring Rules for Continuous Probability Distributions, Management Science 22, 1087–1096.

Matsuda, Kazuhisa, 2004, Introduction to Option Pricing with Fourier Transform: Option Pricing with Exponential Lévy Models, Working Paper.

Melick, William R, and Charles P Thomas, 1997, Recovering an Asset's Implied PDF from Option Prices : An Application to Crude Oil during the Gulf Crisis, The Journal of Financial and Quantitative Analysis 32, 91–115.

Merton, Robert C., 1973, Theory of Rational Option Pricing, Bell Journal of Economics 4, 141–183.

Meyer, Donald J., and Jack Meyer, 2005, Relative risk aversion: What do we know?, Journal of Risk and Uncertainty.

Michailova, Julija, and Ulrich Schmidt, 2016, Overconfidence and Bubbles in Experimental Asset Markets, Journal of Behavioral Finance 17, 280–292.

Mitchell, James, and Stephen G. Hall, 2005, Evaluating, comparing and combining density forecasts using the KLIC with an application to the bank of england and NIESR fan charts of inflation, Oxford Bulletin of Economics and Statistics 67, 995–1033.

Odean, Terrance, 1999, Do investors trade too much?, American Economic Review 89, 1279–1298.

Ortiz-Gracia, Luis, and Cornelis W. Oosterlee, 2013, Robust pricing of european options with wavelets and the characteristic function, SIAM Journal on Scientific Computing 35.

Ortiz-Gracia, Luis, and Cornelis W. Oosterlee, 2016, A highly efficient shannon wavelet inverse fourier technique for pricing European options, SIAM Journal on Scientific Computing 38.

Polkovnichenko, Valery, and Feng Zhao, 2013, Probability weighting functions implied in options prices, Journal of Financial Economics 107, 580–609.

Poon, S H, and C Granger, 2003, Forecasting financial market volatility: A review, Journal of Economic Literature 41, 478–539.

Rebonato, Riccardo, 2004, Volatility and Correlation: The Perfect Hedger and the Fox: Second Edition (John Wiley & Sons, Inc).

Ribeiro, Claudia, and Nick Webber, 2004, Valuing path dependent options in the variance-gamma model by Monte Carlo with a gamma bridge, Journal of Computational Finance 7, 81–100.

Rosenberg, Joshua V., and Robert F. Engle, 2002, Empirical pricing kernels, Journal of Financial Economics 64, 341–372.

Sala, Carlo, Giovanni Barone-Adesi, and Antonietta Mira, 2016, The Impact of Misalignment of Beliefs on the Estimation of the Pricing Kernel SSRN Electronic Journal.

Schmelzle, Martin, 2010, Option pricing formulae using Fourier transform: Theory and application, Preprint, http://pfadintegral. com.

Schobel, Rainer, and Jianwei Zhu, 1999, Stochastic Volatility With an Ornstein-Uhlenbeck Process: An Extension, Review of Finance 3, 23–46.

Schoutens, Wim, Erwin Simons, and Jurgen Tistaert, 2004, A Perfect Calibration! Now What?, Willmott Magazine, 66–78.

Schreindorfer, David, 2013, Tails, Fears, and Equilibrium Option Prices, SSRN Electronic Journal.

Sepp, Artur, 2003, Fourier Transform for Option Pricing under Affine Jump-Diffusions: An Overview, SSRN Electronic Journal.

Shackleton, Mark B., Stephen J. Taylor, and Peng Yu, 2010, A multi-horizon comparison of density forecasts for the S&P 500 using index returns and option prices, Journal of Banking and Finance 34, 2678–2693.

Shefrin, Hersh, 2008, A Behavioral Approach to Asset Pricing. Second Edition Academic Press.

Shen, Junyan, Jianfeng Yu, and Shen Zhao, 2017, Investor sentiment and economic forces, Journal of Monetary Economics 86, 1–21.

Silva, E. M. d. S., and K. H. Kahl, 1993, Reliability of soybean and corn option-based probability assessments, Journal of Futures Markets 13, 765–779.

Silverman, B. W., 1996, Density Estimation for Statistics and Data Analysis.

Simon, David P., and Roy A. Wiggins, 2001, S&P futures returns and contrary sentiment indicators, Journal of Futures Markets 21, 447–462.

Smales, L. A., 2017, The importance of fear: investor sentiment and stock market returns, Applied Economics 49, 3395–3421.

Stambaugh, Robert F., Jianfeng Yu, and Yu Yuan, 2012, The short of it: Investor sentiment and anomalies, Journal of Financial Economics 104, 288–302.

Statman, Meir, Steven Thorley, and Keith Vorkink, 2006, Investor overconfidence and trading volume, Review of Financial Studies.

Taylor, Stephen J., Pradeep K. Yadav, and Yuanyuan Zhang, 2010, The information content of implied volatilities and model-free volatility expectations: Evidence from options written on individual stocks, Journal of Banking and Finance 34, 871–881.

Thomas, P. J., 2016, Measuring risk-aversion: The challenge, Measurement: Journal of the International Measurement Confederation 79, 285–301.

Whaley, Robert E., 2000, The investor fear gauge: Explication of the CBOE VIX, Journal of Portfolio Management 26, 12–17.

Yan, Hongjun, 2010, Is Noise Trading Cancelled Out by Aggregation?, Management Science 56, 1047–1059.

Yatchew, Adonis, and Wolfgang Härdle, 2006, Nonparametric state price density estimation using constrained least squares and the bootstrap, Journal of Econometrics 133, 579–599.

Yu, Jianfeng, and Yu Yuan, 2011, Investor sentiment and the mean-variance relation, Journal of Financial Economics 100, 367–381.

Yun, Jaeho, 2014, Out-of-sample density forecasts with affine jump diffusion models, Journal of Banking and Finance 47, 74–87.

Zhu, Jianwei, 2010, Applications of Fourier Transform to Smile Modeling: Theory and Implementation (Springer Finance).