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# Monte Carlo simulation for investment portfolio rebalancing

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## ABSTRACT

### **Abstract en español:**

El propósito de este trabajo es estudiar la validez de los métodos de simulación Monte Carlo para la valoración de estrategias de rebalanceo de carteras de inversión. Los resultados demuestran la validez del enfoque, que se muestra consistente a lo largo de los distintos experimentos llevados a cabo con carteras de diferentes tamaños, suministrando evidencias de conocidos efectos como la diversificación o el salto de volatilidad.

### **Abstract in English:**

The aim of this research is to address the suitability of the Monte Carlo methods for the valuation of investment portfolio rebalancing strategies. The results confirm the accuracy of this approach, which turns to be consistent throughout the different simulation experiments carried out with various sizes of portfolios, showing evidences of the existence of well-known effects, like the diversification phenomenon or the volatility pumping effect.





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## INTRODUCTION

**F**inancial theory is one of the fields where stochastic calculus offers more remarkable applications. Thus, modelling the price of financial assets using stochastic differential equations is a common practice, supplying a suitable framework in order to analyze a big number of financial problems and accomplish diverse financial decisions.

It is worthwhile mentioning the widely extended use of stochastic calculus in order to value financial derivatives in general and financial options in particular [1]. A financial option is a contract where the holder pays an initial premium in order to get the right, but not the obligation, to buy (if it is a call option) or sell (if it is a put option) a certain asset (the underlying asset), in a certain date (the expiration date), and for a given price (the exercise price). If this right can only be exercised in the expiration date, it is called a European option; if the right can be exercised in any moment until that expiration date, it is called an American option. One of the most famous results in this field is the Black-Scholes formula used to value European options [2].

This methodology has also been successfully applied to evaluate investment decisions, taking into account the uncertainty characterizing the financial environment [3]. In addition, investors tend to distribute their investment budget among various assets building an investment portfolio. If the investor chooses carefully those assets so that they hold different characteristics, the investor can reduce the aggregated risk in a process called diversification [4].

Consequently, optimal portfolio selection becomes a central key of concern for professional investors, who need to constantly readjust or rebalance their portfolio as a response to the

variations in market prices, in order to preserve the consonance between the portfolio structure and the preferences of the investor regarding risk and expected return [5]. So, this methodology results particularly useful for managing investment funds, which have become a very popular investment vehicle in recent times. Although there is a large variety of investment funds, exchange traded funds are a particularly famous subset of these products, since they are transacted in exchange markets, as happens with stocks, and they usually path a stock index [6].

Monte Carlo methods are widely used for several financial purposes, including the valuation of an investment project or a financial instrument. Thus, they are useful when valuing a financial option, because they allow simulating a number of paths for the price of the underlying asset. Also, regarding portfolio valuation, these methods can simulate the behavior of the uncertainty factors and, as a consequence, the investor can achieve a statistical description of the portfolio.

This research is organized as follows. In chapter 2, a description of several strategies for portfolio rebalancing is provided, and these strategies are also mathematically characterized. In chapter 3, the basics of the Monte Carlo methods for financial purposes are discussed and presented. In chapter 4, a description of the data used for this research is provided, together with some basic summary statistics. In chapter 5, the code of the program written in order to carry out the simulations is exposed. In chapter 6, the numerical results of the accomplished simulations are reported and explained. Finally, a set of conclusions driven out from the present research and some tips for further investigation are disclosed.

## INVESTMENT PORTFOLIO REBALANCING

Portfolio rebalancing describes any strategy carried out by investors and portfolio managers consisting in readjusting the portfolio composition by selling units of certain assets in order to buy units of other ones. This results in a transfer of funds among assets inside the portfolio, modifying their particular weights with the purpose of realigning the portfolio with the desired asset allocation. When practiced successfully, portfolio rebalancing can lead to a superior aggregated return [7]. In order to conquer this objective, portfolio rebalancing can be accomplished periodically or occasionally, and it is always justified when its benefits are higher than the costs [8].

Academics and practitioners have proposed several rebalancing strategies, and there is some investigation comparing their performance [9]. In this research, we focus on five different scenarios connected with portfolio rebalancing: constant proportions portfolio, constant proportion portfolio insurance, leveraged exchange traded funds, covered call writing and stop-loss.

## 2.1. Constant proportions portfolio

Considering a portfolio formed by several different investment assets, a constant proportions strategy implies that the proportion of the investment compromised in each asset remains constant through time. This strategy requires an initial investment and from that moment on it is self-financing. At each moment of time, the investor can sell some units of certain assets in order to free sufficient resources to acquire additional units of some other assets. Acting that way, the investor rebalances the portfolio in order to preserve the constant proportion imperative.

The basis of the constant proportions strategy is the Kelly rule [10]. This well-known criterion has been reported to be useful for portfolio optimization [11,12]. Following Forsyth [13], we begin by considering an investor who builds a diversified portfolio with a set of  $n$  different investment assets. Representing each one of those assets by  $S_i; i = 1, \dots, n$ , we suppose that their behavior can be modeled as a Geometric Brownian Motion:

$$dS_i = \mu S_i dt + \sigma S_i dZ_i \quad (1)$$

In this expression,  $\mu$  is a constant representing the drift term,  $\sigma$  is a constant representing the volatility, and  $dZ_i$  is a Wiener process.

At time  $t$ , the investor establishes the investing strategy by deciding the number of units of each asset taking part in the portfolio:  $n_i(t)$ . Considering that this quantity is a function of time, the investor has the ability to rebalance the portfolio in each period.

The market value of the portfolio in a certain period of time can be calculated as follows:

$$P(t) = \sum_{i=1}^n n_i(t) S_i(t) \quad (2)$$

We define the natural log of the value of the portfolio:  $G = \ln P(t)$ . Using the multidimensional version of Ito's Lemma:

$$dG = \left( \frac{\partial G}{\partial t} + \sum_{i=1}^n \frac{\partial G}{\partial S_i} \mu_i S_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 G}{\partial S_i \partial S_j} \sigma_i S_i \sigma_j S_j \rho_{ij} \right) dt + \sum_{i=1}^n \frac{\partial G}{\partial S_i} \sigma_i S_i dZ_i \quad (3)$$

In this expression,  $\rho_{ij}$  is the correlation coefficient between  $dZ_i$  and  $dZ_j$ . Given that  $G = \ln P(t)$ , it follows:

$$\frac{\partial G}{\partial t} = 0; \quad \frac{\partial G}{\partial S_i} = \frac{n_i}{P}; \quad \frac{\partial^2 G}{\partial S_i \partial S_j} = -\frac{n_i n_j}{P^2} \quad (4)$$

Using (4) in (3):

$$dG = \left( \sum_{i=1}^n \frac{n_i}{P} \mu_i S_i - \frac{1}{2} \sum_{i,j=1}^n \frac{n_i n_j}{P^2} \sigma_i S_i \sigma_j S_j \rho_{ij} \right) dt + \sum_{i=1}^n \frac{n_i}{P} \sigma_i S_i dZ_i \quad (5)$$

If the investor rebalances the portfolio every time in order to apply a constant proportions strategy, the weight of each of the assets is calculated as the proportion that the value invested in the asset represents into the whole value of the portfolio:

$$\omega_i = \frac{n_i S_i}{P(t)}; \sum_{i=1}^n \omega_i = 1 \quad (6)$$

It can be noted that  $\omega_i$  does not depend on  $t$ , since we assume that the investor can rebalance the portfolio continuously. Using (6) in (5):

$$dG = \left( \sum_{i=1}^n \omega_i \mu_i - \frac{1}{2} \sum_{i,j=1}^n \omega_i \omega_j \sigma_i \sigma_j \rho_{ij} \right) dt + \sum_{i=1}^n \omega_i \sigma_i dZ_i \quad (7)$$

The solution to this equation is:

$$G(t) = G(0) + \left( \sum_{i=1}^n \omega_i \mu_i - \frac{1}{2} \sum_{i,j=1}^n \omega_i \omega_j \sigma_i \sigma_j \rho_{ij} \right) t + \sum_{i=1}^n \omega_i \sigma_i (Z_i(t) - Z_i(0)) \quad (8)$$

Considering that  $G = \ln P(t)$ , it follows:

$$\ln \left( \frac{P(t)}{P(0)} \right) = \left( \sum_{i=1}^n \omega_i \mu_i - \frac{1}{2} \sum_{i,j=1}^n \omega_i \omega_j \sigma_i \sigma_j \rho_{ij} \right) t + \sum_{i=1}^n \omega_i \sigma_i (Z_i(t) - Z_i(0)) \quad (9)$$

Thus, the mean is:

$$E \left( \ln \left( \frac{P(t)}{P(0)} \right) \right) = \left( \sum_{i=1}^n \omega_i \mu_i - \frac{1}{2} \sum_{i,j=1}^n \omega_i \omega_j \sigma_i \sigma_j \rho_{ij} \right) t \quad (10)$$

And the variance is:

$$\text{Var} \left( \ln \left( \frac{P(t)}{P(0)} \right) \right) = \left( \sum_{i,j=1}^n \omega_i \omega_j \sigma_i \sigma_j \rho_{ij} \right) t \quad (11)$$

If we consider the particular case when all the assets of the portfolio have the same drift and the same volatility:

$$\mu_i = \mu; \sigma_i = \sigma; \omega_i = \frac{1}{n}; \forall i = 1, \dots, n \quad (12)$$

Using (12) in (8):

$$G(t) = G(0) + \left( \mu - \frac{\sigma^2}{2n^2} \sum_{i,j=1}^n \rho_{ij} \right) t + \frac{\sigma}{n} \sum_{i=1}^n (Z_i(t) - Z_i(0)) \quad (13)$$

Given that  $G = \ln P(t)$ , it follows:

$$\ln \left( \frac{P(t)}{P(0)} \right) = \left( \mu - \frac{\sigma^2}{2n^2} \sum_{i,j=1}^n \rho_{ij} \right) t + \frac{\sigma}{n} \sum_{i=1}^n (Z_i(t) - Z_i(0)) \quad (14)$$

Therefore, the mean is:

$$E \left( \ln \left( \frac{P(t)}{P(0)} \right) \right) = \left( \mu - \frac{\sigma^2}{2n^2} \sum_{i,j=1}^n \rho_{ij} \right) t = \left( \mu - \frac{\sigma^2}{2} \left( \frac{1}{n^2} \sum_{i,j=1}^n \rho_{ij} \right) \right) t \quad (15)$$

And the variance is:

$$\text{Var} \left( \ln \left( \frac{P(t)}{P(0)} \right) \right) = \left( \sigma^2 \left( \frac{1}{n^2} \sum_{i,j=1}^n \rho_{ij} \right) \right) t \quad (16)$$

Note that in (15) and (16):

$$\frac{1}{n^2} \sum_{i,j=1}^n \rho_{ij} \leq 1 \quad (17)$$

Now, we compare the variance of the portfolio with the variance of a single asset following a Geometric Brownian Motion:

$$\text{Var} \left( \ln \left( \frac{P(t)}{P(0)} \right) \right) \leq \text{Var} \left( \ln \left( \frac{S(t)}{S(0)} \right) \right) = \sigma^2 t \quad (18)$$

Consequently, the variance of the portfolio is lower than the one observed in the single asset. This result is a consequence of the diversification effect inherent to the portfolio building process.

Also, we compare the mean of this portfolio with the mean of a single asset following a Geometric Brownian Motion:

$$E\left(\ln\left(\frac{P(t)}{P(0)}\right)\right) \geq E\left(\ln\left(\frac{S(t)}{S(0)}\right)\right) = \left(\mu - \frac{\sigma^2}{2}\right)t \quad (19)$$

So, the drift of this portfolio is higher than the drift of the single asset, even in the case when all the assets included in the portfolio have the same drift. This greater drift observed in the portfolio is usually referred as volatility pumping, because the value directly depends on the volatility. The result is a consequence of the rebalancing strategy, allowing the investor to apply the basic principle of “buy low and sell high” every time that the portfolio is rebalanced, in order to preserve the constant proportions rule. So, this effect appears in every constant proportions strategy, given that the investor incorporates a minimum of two assets in the portfolio, and the fluctuations of prices over time are not identical from one asset to another [14].

## 2.2. Constant proportion portfolio insurance

The constant proportion portfolio insurance is a strategy consisting on preserving the risk exposure of the portfolio on a certain value established as a multiple of the excess of the return over a minimum return level [15].

Suppose an investor that distributes the investment budget between a risky asset and a risk-free asset (a government bond, for example). This means that the investor is building a portfolio with only two assets. Consider that the drift of the risky asset is  $\mu$  and its volatility is  $\sigma$ . Also, the drift of the risk-free asset is the risk-free rate  $r$ , and it has no volatility. Finally, the correlation coefficient between the two assets is null, since the risk-free asset is always uncorrelated with any risky asset.

If we establish that  $\omega_1$  is the weight of the risky asset in the portfolio, and  $\omega_2$  is the weight of the risk-free asset, it follows that:

$$\omega_1 + \omega_2 = 1 \Rightarrow \omega_2 = 1 - \omega_1 \quad (20)$$

According to (8), the mean of this portfolio is:

$$E\left(\ln\left(\frac{P(t)}{P(0)}\right)\right) = \left(r + \omega_1(\mu - r) - \frac{\omega_1^2\sigma^2}{2}\right)t \quad (21)$$

In addition, the variance of this portfolio depends only on the risky asset:

$$\text{Var}\left(\ln\left(\frac{P(t)}{P(0)}\right)\right) = (\omega_1^2 \sigma^2)t \quad (22)$$

We use  $S$  to represent the price of the risky asset, and  $B$  to represent the price of the risk-free asset. If we consider that  $n_S$  is the number of units of the risky asset taking part in the portfolio, and  $n_B$  is the number of units of the risk-free asset, the value of the portfolio at any time  $t_i$  is:

$$P(t_i) = n_S(t_{i-1}) \cdot S(t_i) + n_B(t_{i-1}) \cdot B(t_{i-1}) \cdot e^{r(t_i-t_{i-1})} \quad (23)$$

In this type of portfolio, the investor needs to establish two additional parameters. First, the investor establishes a floor, that is, the minimum guaranteed return the investor wishes to get from the portfolio. Second, the investor establishes a multiplier, that is, a coefficient that collects the investor's risk profile, presenting higher values for more aggressive investors.

If we use  $F$  to represent the floor, and  $M$  to represent the multiplier, when the investor is rebalancing the portfolio at a certain time  $t_i$ , the units invested in the risky asset are:

$$n_S(t_i) = M(t_i) \left( \max\left(\frac{P(t_i)}{S(t_i)} - F(t_i), 0\right) \right) \quad (24)$$

And the units invested in the risk-free asset are:

$$n_B(t_i) = \frac{n_B(t_{i-1}) \cdot B(t_{i-1}) \cdot e^{r(t_i-t_{i-1})} - (n_S(t_i) - n_S(t_{i-1}))S(t_i)}{B(t_i)} \quad (25)$$

As a result, each time the portfolio is rebalanced, the investor assures that the risk exposure remains at the level determined by the multiplier, investing more in the risky asset as the value of the portfolio rises above the floor, and investing more in the risk-free asset when the value of the portfolio decreases.

### 2.3. Leveraged exchange traded funds

Leveraged exchange traded funds are fresh, popular and novel investment instruments that allow an investor to obtain a certain multiple of the daily return of the reference index [16]. The most common multiples are 2x, 3x and 5x, representing the double, triple and quintuple of the daily return. In these funds, once the investor has placed the investment compromise, the fund manager borrows the complementary amount required to obtain the offered return. Thus, for each monetary unit compromised by the investor, the fund manager borrows one additional monetary unit in a 2x fund, two additional monetary units in a 3x fund, and 4 additional

monetary units in a 5x fund. Consequently, these products can be considered a special case of a constant proportion portfolio insurance product with the following particularities:

- $F(t_i) = F = 0$ , that is, the minimum guaranteed return is null and remains constant from one period to another.
- $M(t_i) = M > 1$ , that is, the multiplier allows the investor to get a multiple of the daily return, and remains constant from one period to another.
- $n_B(t_i) < 0$ , that is, the number of units of the risk-free asset is negative, which represents borrowing or negative investing.

As a result, using expression (24), now the number of units invested in the risky asset can be calculated as follows:

$$n_s(t_i) = M \left( \max \left( \frac{P(t_i)}{S(t_i)}, 0 \right) \right) \quad (26)$$

There are two versions of these leveraged exchange traded funds. On one hand, we have the bull version, which allows the investor to get a multiple of the positive daily return of the reference index, that is, in expressions (21) and (22),  $\omega_1$  has a positive value. And, on the other hand, there is the bear version, which allows the investor to obtain a multiple of the negative daily return of the reference index, that is, in expressions (21) and (22),  $\omega_1$  has a negative value.

## 2.4. Covered call writing

The covered call writing strategy consists on selling call options on owned assets, with a strike price just above its current market value, and given that the number of calls is not higher than the number of assets owned. This is a usual rebalancing strategy carried out by fund managers, since it allows getting a superior return due to the call premium when the prospects anticipate a stagnation or decrease of the underlying asset price in the short term [17].

Following Forsyth [13], we begin by considering the well-known put-call parity driven out from the Black-Scholes formula to value a European option [2]:

$$C(t) - P(t) = S - Ke^{-r(T-t)} \quad (27)$$

In this expression,  $C(t)$  represents the price of the call option, and  $P(t)$  stands for the price of the put option, both on the same underlying asset, with the same strike price, and the same expiration date.

Rearranging in (27), we get:

$$S - C(t) = Ke^{-r(T-t)} - P(t) \quad (28)$$

Note that  $S - C(t)$  represents the covered call writing strategy: owning a stock whose market value is  $S$  and selling a call option. According to (28), this is equivalent to sell a put and invest the amount  $Ke^{-r(T-t)}$  in a risk-free asset, which allows the investor to obtain the quantity  $K$  at the expiry date with no risk exposure. So, under these circumstances, the put option will never be exercised and, as a consequence, the investor obtains an extra profit derived from the option premium.

## 2.5. Stop-loss

The stop-loss strategy consists on establishing a floor price for the value of an owned asset, so that, if the market value of the asset falls below the floor, a sell order is submitted to the market. Then, the investor takes the amount resulting from the sale and puts it into a risk-free asset during the remaining investment period. This is referred as the pure stop-loss strategy, and it is another common rebalancing strategy used in order to provide some sort of insurance by establishing a limit for losses and, as a result, limiting the risk of a downturn in the portfolio value. There is another version of this strategy called stop-loss start-gain. In this variation, the sell order is combined with a buy order, so that once the asset has been sold under the same circumstances described for the pure stop-loss strategy, a new buy order is placed and it will be executed if the market value of the asset rises over the floor price. Acting this way, the portfolio can still benefit from the rise potential [18].

The hedging result of this strategy is quite similar to the one obtained when operating with options, using the floor as the strike price. Thus, the stop-loss order can be assimilated with a long position in a put contract, and the start-gain order can be assimilated to a long position in a call contract. The main difference between these comparable strategies relates to their respective costs. Thus, the options strategy has a superior initial cost derived from the premium. However, in the stop-loss strategy there is also a hidden cost, since once the order is submitted to the market, it is executed at the price showing in the order book, which will not match exactly the floor, so that difference constitutes the cost of the operation.

## MONTE CARLO METHODS AND FINANCIAL APPLICATIONS

The Monte Carlo methods constitute a wide set of algorithms extensively used for optimization and numerical integration, based on repetitive random sampling in order to conquer numerical outcomes. Therefore, they are close to one of the most famous results of probability theory, the law of large numbers, which is a theorem stating that once a certain experiment is repeated a large number of times, the average of the results gets close to the expected value, and the approximation turns to be better as the number of repetitions increases [19]. Given a certain random variable ( $X$ ), assuming that a sequence of independent events following its distribution can be generated ( $X_1, X_2, \dots, X_n$ ), and supposing that  $f$  satisfies  $E(|f(X)|) < \infty$ , the following holds:

$$\lim_{n \rightarrow \infty} \left( \left| \frac{1}{n} \sum_{i=0}^n f(X_i) - E(f(X)) \right| \right) = 1 \Rightarrow \frac{1}{n} \sum_{i=0}^n f(X_i) \rightarrow E(f(X)) \quad (29)$$

As a consequence, the error can be computed as follows [20]:

$$\epsilon_n = \frac{1}{n} \sum_{i=0}^n f(X_i) - E(f(X)) \quad (30)$$

Monte Carlo methods are extensively used in finance to simulate the different paths that a certain financial magnitude can follow according to different uncertainty sources framed into a stochastic model. This approach results helpful in analyzing different financial problems regarding financial derivatives valuation, investment decisions and portfolio management [21,22].

## DATA AND SUMMARY STATISTICS

The stock data used in this research come from the official source INFOBOLSA, a company owned by Bolsas y Mercados Españoles (BME) and Deutsche Börse. The sample contains data from seven companies from different industries taking part in the IBEX-35 stock index. The selected companies are:

- TELEFÓNICA (TEL), from the telecommunications industry.
- BBVA (BBV), from the banking industry.
- IBERDROLA (IBE), from the electrical industry.
- ACS (ACS), from the construction industry.
- INDITEX (IND), from the textile industry.
- REPSOL (REP), from the petroleum industry.
- MAPFRE (MAP), from the insurance industry.

For each one of these companies, the daily close price during a 3-year period is collected, resulting in 765 observations for each company, that is, 255 observations per year.

Total observations	765
Time in years	3
Number of steps (total observations per year)	255
Time step between observations in years ( $n$ )	0.00392157

TABLE 4.1. Summary of parameters for the collected data

The last close price of the series is selected as the spot price to be used as the starting point for the simulation ( $S^0$ ):

Company	Spot
TEL	13.750
BBV	9.372
IBE	6.013
ACS	31.400
IND	29.075
REP	18.215
MAP	3.391

TABLE 4.2. Spot price for the selected companies

Representing by  $S_t$  the close price at a certain time  $t$ , the natural log of the stock return at that moment is computed as follows:

$$R_{it} = \ln\left(\frac{S_t}{S_{t-1}}\right); i = 1, \dots, 7; t = 2, \dots, 765 \quad (31)$$

The average ( $\mu_i$ ) and standard deviation ( $\sigma_i$ ) of the return for each company is calculated. In order to accomplish the calibration of the stochastic model for each one of the companies, the volatility is computed as follows:

$$\frac{\sigma_i}{\sqrt{n}}; i = 1, \dots, 7 \quad (32)$$

In addition, the drift is estimated as follows:

$$\frac{\mu_i}{n} + \frac{\sigma_i^2}{2n}; i = 1, \dots, 7 \quad (33)$$

The results of the drift and volatility calibration are shown in Table 4.3:

Company	Volatility	Drift
TEL	0.23460293	0.16022780
BBV	0.30028668	0.30188555
IBE	0.24763297	0.26631424
ACS	0.31587918	0.38850286
IND	0.24543137	0.30221156
REP	0.28789559	0.20376467
MAP	0.32839495	0.26143924

TABLE 4.3. Drift and volatility for the selected companies

In addition, the covariance is also estimated:

$$\frac{\sigma_{ij}}{n}; i, j = 1, \dots, 7 \quad (34)$$

The results are shown in Table 4.4:

	TEL	BBV	IBE	ACS	IND	REP	MAP
TEL	0.055039	0.056962	0.049004	0.044640	0.028665	0.052636	0.049624
BBV	0.056962	0.090172	0.057424	0.059614	0.037961	0.066662	0.067700
IBE	0.049004	0.057424	0.061322	0.049279	0.028233	0.055223	0.053590
ACS	0.044640	0.059614	0.049279	0.099780	0.030779	0.052906	0.062150
IND	0.028665	0.037961	0.028233	0.030779	0.060237	0.035908	0.033889
REP	0.052636	0.066662	0.055223	0.052906	0.035908	0.082884	0.062095
MAP	0.049624	0.067700	0.053590	0.062150	0.033889	0.062095	0.107843

TABLE 4.4. Covariance data for the selected companies

The correlation coefficient is also computed, and the results are provided in Table 4.5:

Correl	TEL	BBV	IBE	ACS	IND	REP	MAP
TEL	1.000000	0.808560	0.843506	0.602378	0.497839	0.779315	0.644113
BBV	0.808560	1.000000	0.772234	0.628476	0.515075	0.771090	0.686526
IBE	0.843506	0.772234	1.000000	0.629988	0.464540	0.774595	0.658986
ACS	0.602378	0.628476	0.629988	1.000000	0.397007	0.581764	0.599137
IND	0.497839	0.515075	0.464540	0.397007	1.000000	0.508189	0.420465
REP	0.779315	0.771090	0.774595	0.581764	0.508189	1.000000	0.656786
MAP	0.644113	0.686526	0.658986	0.599137	0.420465	0.656786	1.000000

TABLE 4.5. Correlation data for the selected companies

Finally, data from the whole IBEX-35 stock index are also collected computing its logarithmic return using expression (31), and the estimation of its drift and its volatility is also carried out using expressions (32) and (33) respectively:

Drift	0.21740117
Volatility	0.18717144

TABLE 4.6. Drift and volatility for the stock index

## VISUAL BASIC CODE FOR MONTE CARLO SIMULATION

For the purpose of this research, a specific program using Visual Basic is deployed. Visual Basic is an event driven programming language developed by Microsoft®, which in combination with the spreadsheet application Excel® has been reported to be a useful tool for financial applications [23].

Focusing on the constant proportions strategy, two different scenarios will be considered: first, the heterogeneous profile scenario, where each one of the stocks involved in the portfolio presents different drift and volatility; and second, the homogeneous profile scenario, where all the stocks taking part in the portfolio have the same drift and volatility.

In this experiment, the spreadsheet is used to store the initial values needed for the simulation. These initial values are:

- Number of steps and time step shown in Table 4.1
- Spot price of the stocks shown in Table 4.2
- Drift and volatility of the stocks shown in Table 4.3
- The number of assets to include in the portfolio and the number of simulations to accomplish, established manually as shown in Figure 5.1:

Assets	7
Iterations	500

FIGURE 5.1. Basic data for simulation

- The initial weigh of each asset in the portfolio, as shown in Figure 5.2. This is computed by making the calculation expressed in (6), once the initial number of units of each asset to include in the portfolio have been manually introduced:

Company	Units	Spot	Weight
TEL	2	13.75	0.06997509
BBV	2	9.372	0.04769502
IBE	2	6.013	0.03060074
ACS	5	31.4	0.39949414
IND	4	29.075	0.29593101
REP	3	18.215	0.13904686
MAP	2	3.391	0.01725713

FIGURE 5.2. Initial weights in the portfolio

The spreadsheet is also the destination of the program outcome, consisting on the average and standard deviation of the simulation results, as shown in Figure 5.3:

Average (Simulation)	0.2300129
Std. Dev. (Simulation)	0.2217585

FIGURE 5.3. Preview of simulation results

As a previous step for the simulation, a program for accomplishing the Cholesky decomposition is built, since it is later used in order to generate correlated random variables from the correlation matrix showed in Table 4.5. The code of this program is the following:

```
Dim matrix() As Double 'Original matrix
Dim lower() As Double 'Lower triangular matrix, result of the Cholesky decomposition
Dim assets As Integer 'Number of assets in the portfolio, that is, number of columns/rows of the original matrix
Dim start_matrix As String 'Top-left corner of the original matrix
Dim start_lower As String 'Top-left corner of the result matrix
Dim element As Double 'Needed for intermediate calculus
```

FIGURE 5.4. Dimensioning variables for Cholesky decomposition

The program begins by dimensioning the required variables for the data and results of the Cholesky decomposition.

```
ReDim matrix(1 To assets, 1 To assets)
ReDim lower(1 To assets, 1 To assets)
```

FIGURE 5.5. Resizing variables for Cholesky decomposition

Next, the multidimensional variables used to store matrix data are resized.

```
assets = Range("B5").Value|
start_matrix = "E12"
start_lower = "E21"
Range("E21:K27").ClearContents 'Clean space of the resulting matrix
```

FIGURE 5.6. Feeding variables for Cholesky decomposition

The program gets basic data referred to the spreadsheet, and it also cleans up the space where the result is going to be located.

```
For i = 1 To assets
  For j = 1 To assets
    matrix(i, j) = Range(start_matrix).Offset(i - 1, j - 1).Value
    lower(i, j) = 0
  Next j
Next i
```

FIGURE 5.7. Feeding matrix for Cholesky decomposition

This loop gets the correlation matrix data from the spreadsheet and it initially assigns null value to all the elements of the resulting matrix.

```
For i = 1 To assets
  For j = 1 To assets
    element = matrix(i, j)
    For k = 1 To i - 1
      element = element - lower(i, k) * lower(j, k)
    Next k
    If i = j Then
      lower(i, j) = Sqr(element)
    ElseIf i < j Then
      lower(j, i) = element / lower(i, i)
    End If
  Next j
Next i
```

FIGURE 5.8. Calculus for Cholesky decomposition

This piece of code shows some nested loops that make the necessary calculations in order to get the Cholesky decomposition result.

```
For i = 1 To assets
  For j = 1 To assets
    Range(start_lower).Offset(i - 1, j - 1).Value = lower(i, j)
  Next j
Next i
```

FIGURE 5.9. Transferring results of Cholesky decomposition

Finally, the results of the Cholesky decomposition are transferred to the spreadsheet, as shown in Figure 5.10:

Cholesky (L)	TEL	BBV	IBE	ACS	IND	REP	MAP
TEL	1	0	0	0	0	0	0
BBV	0.8085603	0.5884133	0	0	0	0	0
IBE	0.8435062	0.153308	0.5147756	0	0	0	0
ACS	0.6023782	0.2403357	0.1651834	0.7430301	0	0	0
IND	0.4978385	0.1912638	0.0296975	0.062241	0.8431009	0	0
REP	0.7793151	0.2395709	0.1563974	0.0389077	0.0798577	0.550384	0
MAP	0.6441133	0.2816412	0.1408281	0.1617519	0.0375797	0.1017943	0.6693473

Cholesky (U)	TEL	BBV	IBE	ACS	IND	REP	MAP
TEL	1	0.8085603	0.8435062	0.6023782	0.4978385	0.7793151	0.6441133
BBV	0	0.5884133	0.153308	0.2403357	0.1912638	0.2395709	0.2816412
IBE	0	0	0.5147756	0.1651834	0.0296975	0.1563974	0.1408281
ACS	0	0	0	0.7430301	0.062241	0.0389077	0.1617519
IND	0	0	0	0	0.8431009	0.0798577	0.0375797
REP	0	0	0	0	0	0.550384	0.1017943
MAP	0	0	0	0	0	0	0.6693473

FIGURE 5.10. Results of the Cholesky decomposition

### 5.1. Heterogeneous profile scenario

In this section, the program used to simulate the constant proportions strategy under the heterogeneous profile scenario is reported.

```
Dim number() As Double 'Units of each asset in the portfolio in each rebalancing
Dim drift() As Double 'Drift of each asset
Dim volatility() As Double 'Volatility of each asset
Dim weight() As Double 'Constant weight of each asset in the portfolio
Dim asset() As Double 'Value of each asset in each step
Dim random() As Double 'Random number
Dim correlated() As Double 'Correlated random numbers
Dim portfolio() As Double 'Total value of the portfolio
Dim sum As Double 'Needed to compute average log return of the portfolio
Dim sum_sq As Double 'Needed to compute standard deviation of log return of the portfolio
```

FIGURE 5.11. Multidimensional variables in the heterogeneous scenario

The code begins defining the multidimensional variables needed for the latter simulation.

```
'These variables are needed to signal some key positions in the worksheet
Dim start_spot As String 'List of spot prices
Dim start_units As String 'List of units
Dim start_drift As String 'List of drifts
Dim start_volatility As String 'List of standar deviations
Dim start_weight As String 'List of weights
Dim start_cholesky As String 'Result of Cholesky decomposition

start_spot = "P2"
start_units = "Q2"
start_drift = "N2"
start_volatility = "M2"
start_weight = "R2"
start_cholesky = "E30"
```

FIGURE 5.12. Positional variables in the heterogeneous scenario

Next, the positional variables are defined, used to signal some key positions in the spreadsheet, and they are also fed with their initial values.

```

Dim assets As Integer 'Total assets in the portfolio
Dim step As Double 'Size of the time step
Dim iterations As Long 'Total number of iterations
Dim steps As Integer 'Total number of steps
Dim pi As Double 'Needed for the Box-Muller algorithm

assets = Range("B5").Value
step = Range("B4").Value
iterations = Range("B6").Value
steps = Range("B3").Value
pi = 3.14159265358979

```

FIGURE 5.13. Unidimensional variables in the heterogeneous scenario

Some unidimensional variables are also defined and their initial values are immediately assigned.

```

ReDim number(1 To iterations, 0 To steps, 1 To assets)
ReDim drift(1 To assets)
ReDim volatility(1 To assets)
ReDim weight(1 To assets)
ReDim asset(1 To iterations, 0 To steps, 1 To assets)
ReDim random(1 To iterations, 1 To steps, 1 To assets)
ReDim correlated(1 To iterations, 1 To steps, 1 To assets)
ReDim portfolio(0 To iterations, 0 To steps)

```

FIGURE 5.14. Resizing multidimensional variables in the heterogeneous scenario

The multidimensional variables previously defined are now resized according to their respective requirements.

```

For i = 1 To assets 'Get initial values
    drift(i) = (Range(start_drift).Offset(i - 1).Value - Range(start_volatility).Offset(i - 1).Value ^ 2 / 2) * step
    volatility(i) = Range(start_volatility).Offset(i - 1).Value * Sqr(step)
    weight(i) = Range(start_weight).Offset(i - 1).Value
Next i

```

FIGURE 5.15. Collecting stock parameters in the heterogeneous scenario

This loop collects the drift, volatility and weight of each asset taking part in the portfolio from the spreadsheet.

```

For i = 1 To iterations 'Each iteration
    portfolio(0, 0) = Range("Q9").Value 'Get initial value of the portfolio

```

FIGURE 5.16. Iterations-loop in the heterogeneous scenario

The main loop of the code, which will make one execution by iteration, is defined and it starts by collecting the initial value of the portfolio.

```

For k = 1 To assets 'Get initial values for each asset
    asset(1, 0, k) = Range(start_spot).Offset(k - 1).Value
    number(1, 0, k) = Range(start_units).Offset(k - 1).Value
Next k

```

FIGURE 5.17. Collecting stock values in the heterogeneous scenario

This next loop collects the initial spot price and number of units for each asset included in

the portfolio.

```
For j = 1 To steps 'Each step
```

FIGURE 5.18. Steps-loop in the heterogeneous scenario

A secondary loop is initialized in order to be executed for each time step.

```
For k = 1 To assets 'Each asset
  random(i, j, k) = Sqr((-2 * Log(Rnd)) * Sin(2 * pi * Rnd) 'Generate random numbers (Box-Muller)
  For l = 1 To assets 'Go through Cholesky decomposition in order to get correlated random numbers
    correlated(i, j, k) = correlated(i, j, k) + random(i, j, l) * Range(start_cholesky).Offset(l - 1, k - 1).Value
  Next l
  'Compute the value of the asset
  asset(i, j, k) = asset(i, j - 1, k) * Exp(drift(k) + volatility(k) * correlated(i, j, k))
  'Add the asset to the portfolio value
  portfolio(i, j) = portfolio(i, j) + asset(i, j, k) * number(i, j - 1, k)
Next k
```

FIGURE 5.19. Random variation in the heterogeneous scenario

In this loop, four different actions are executed for each asset in the portfolio:

- The Box-Muller transform is used to generate a set of independent random numbers.
- The results of the Cholesky decomposition are used to transform the previous set of independent random numbers into a set of correlated random numbers.
- Using the correlated random numbers, the next spot price for each of the assets is generated.
- The generated prices for the assets are aggregated to get the new value of the portfolio.

```
For k = 1 To assets 'Rebalancing
  number(i, j, k) = weight(k) * portfolio(i, j) / asset(i, j, k)
Next k
```

FIGURE 5.20. Portfolio rebalancing in the heterogeneous scenario

Using the new spot price for each asset, and the new portfolio value, the number of units to include in the portfolio in order to preserve the constant proportions premise is calculated.

```
Next j
'In order to compute the average
sum = sum + Log(portfolio(i, steps) / portfolio(0, 0))
'In order to compute the standard deviation
sum_sq = sum_sq + Log(portfolio(i, steps) / portfolio(0, 0)) ^ 2
Next i
```

FIGURE 5.21. Aggregating simulation results in the heterogeneous scenario

The simulation results are aggregated in order to compute the results at the end of the program.

```
Range("B8") = sum / iterations 'Average  
Range("B9") = Sqr(sum_sq / iterations - Range("b8").Value ^ 2) 'Standard deviation
```

FIGURE 5.22. Transferring simulation results in the heterogeneous scenario

Finally, the average and standard deviation of the simulation results are calculated and transferred to the spreadsheet.

## 5.2. Homogeneous profile scenario

The homogenous profile scenario and the heterogeneous profile scenario share the most part of the data and the code. So, this section reports the main differences compared to the previous section.

When it comes to the initial data, the only difference concerns the drift, the volatility and the weights of the stocks included in the portfolio. The values estimated for the IBEX-35 stock index and reported in Table 4.5 are used for all the stocks.

Regarding the code, the main difference is shown in Figure 5.23:

```
drift = (Range(start_drift).Value - Range(start_volatility).Value ^ 2 / 2) * step  
volatility = Range(start_volatility).Value * Sqr(step)  
weight = Range(start_weight).Value
```

FIGURE 5.23. Collecting stock values in the homogeneous scenario

When collecting the initial stock values, instead of the loop showed in Figure 5.17, the values are collected in unidimensional variables in order to use the same values for all the stocks.

## RESULTS OF THE MONTE CARLO SIMULATION

In this section, the main results driven out from the accomplished simulations are reported. A separate analysis for the heterogeneous profile scenario and the homogeneous profile scenario is presented.

### 6.1. Heterogeneous profile scenario

As a first approximation, six portfolios with different sizes are constructed. For each of these portfolios, the mean and standard deviation is analytically computed for comparison purposes using expressions (10) and (11), respectively.

Assets	Mean	Std. Dev.
2	0.20810834	0.2601308
3	0.22038214	0.24600959
4	0.25352598	0.24411147
5	0.26103494	0.22665497
6	0.24491504	0.23039421
7	0.24302839	0.23332484

TABLE 6.1. Mean and standard deviation in the heterogeneous profile scenario

A graph with this data is provided in order to get a better approximation:

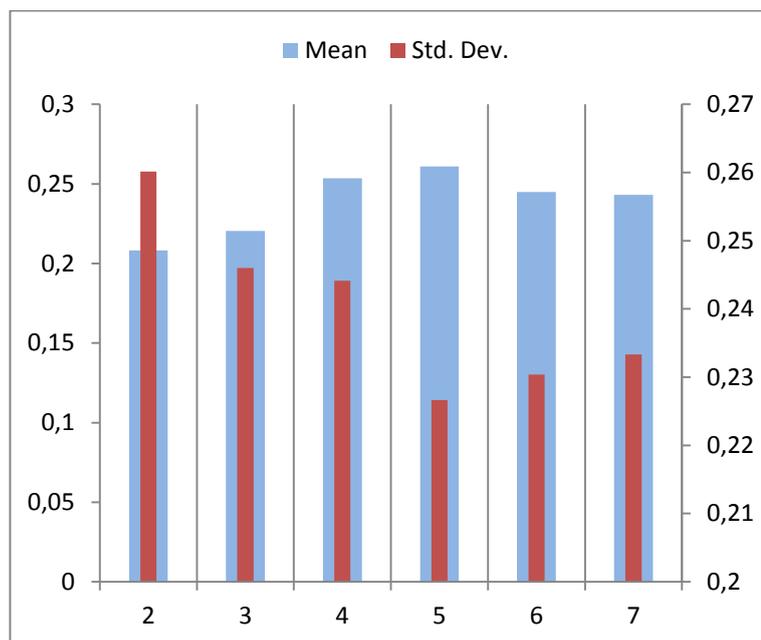


FIGURE 6.1. Mean and standard deviation in the heterogeneous profile scenario

As the number of assets taking part in the portfolio increases, the average return tends to get higher, as a result of the “volatility pumping” effect. In addition, the standard deviation tends to get lower, as a consequence of the diversification effect.

Focusing on the 3-asset portfolio, several simulation experiments are carried out, with the following results:

Iterations	Mean		Std. Dev.	
	Simulation	Error	Simulation	Error
500	0.23253333	1.215119E-02	0.26336331	1.735373E-02
750	0.22923937	8.857230E-03	0.25653174	1.052216E-02
5,000	0.2164948	3.887349E-03	0.24890342	2.893835E-03
30,000	0.21960175	7.803976E-04	0.24565907	3.505185E-04
50,000	0.21988049	5.016585E-04	0.24586863	1.409561E-04

TABLE 6.2. 3-asset portfolio simulation in the heterogeneous profile scenario

These data show that the Monte Carlo simulation is a suitable methodology for the purpose of this research, as it approximates well to the analytical result, and the error tends to decrease as the number of iterations increases. These data are represented in the following graph:

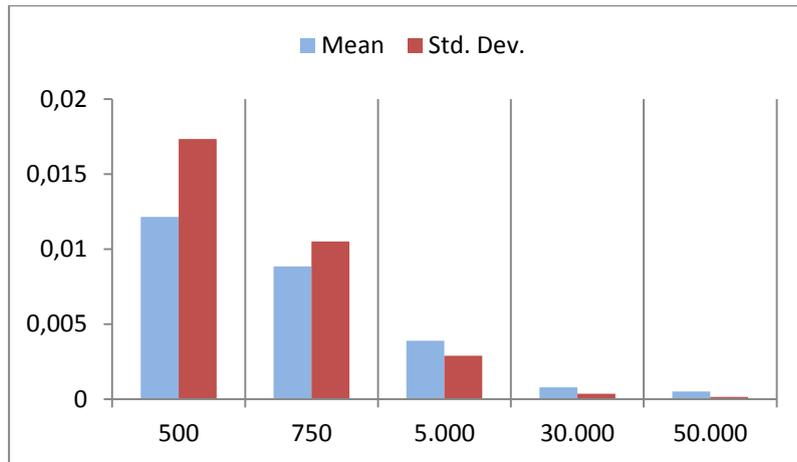


FIGURE 6.2. 3-asset portfolio simulation in the heterogeneous profile scenario

Similar simulation experiments are performed with the other defined portfolios, in order to explore how the Monte Carlo methodology behaves with portfolios of different sizes. The following set of graphs represents the mean value for the first 500 iterations of the simulation experiment:

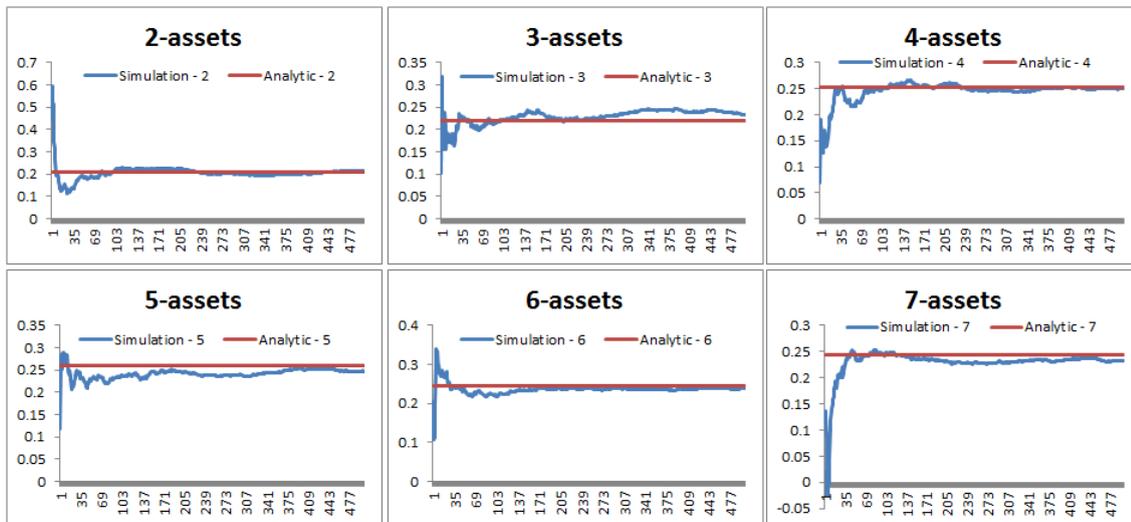


FIGURE 6.3. n-asset portfolio simulation in the heterogeneous profile scenario

These graphs show that the algorithm needs a larger number of iterations to converge around the exact value as the number of assets included in the portfolio increases, as a consequence of the higher uncertainty introduced in the portfolio.

## 6.2. Homogeneous profile scenario

Following a similar approach that the one exposed in the previous scenario, the analytical solution for the mean and the standard deviation is computed for comparison purposes, using expressions (14) and (15), respectively:

Assets	Mean	Std. Dev.
2	0.16580183	0.2067347
3	0.16656308	0.20301903
4	0.16860542	0.19269677
5	0.1708002	0.18094885
6	0.17066271	0.18170711
7	0.17102422	0.17970656

TABLE 6.3. Mean and standard deviation in the homogeneous profile scenario

The “volatility pumping” effect and the diversification effect arise again, even more clearly than in the previous scenario, as it can be seen in the next graph:

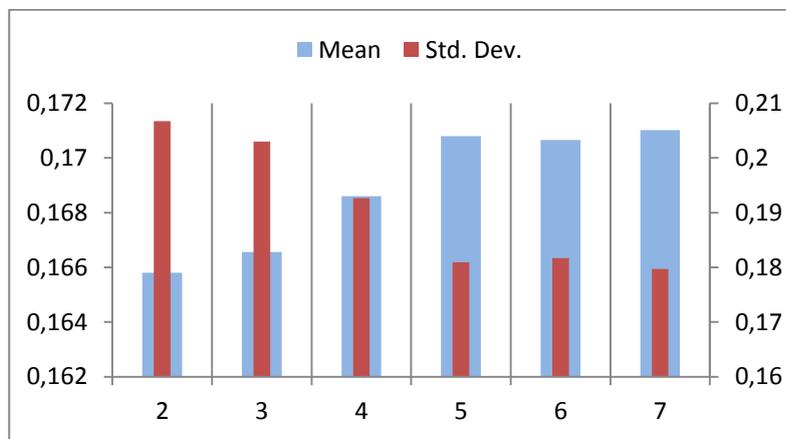


FIGURE 6.4. Mean and standard deviation in the homogeneous profile scenario

Turning the sights towards the 3-asset portfolio, the simulation experiments deliver the following results:

Iterations	Mean		Std. Dev.	
	Simulation	Error	Simulation	Error
750	0.14811498	1.844810E-02	0.20657221	3.553183E-03
1,500	0.15782783	8.735248E-03	0.20595867	2.939642E-03
7,500	0.16754441	9.813301E-04	0.2008839	2.135128E-03
15,000	0.16579539	7.676877E-04	0.20117947	1.839559E-03
50,000	0.16667656	1.134770E-04	0.20263181	3.872231E-04

TABLE 6.4. 3-asset portfolio simulation in the homogeneous profile scenario

The Monte Carlo simulation provides a suitable estimation of the exact result, and the error decreases as the number of iterations increases, as it is shown in this graph:

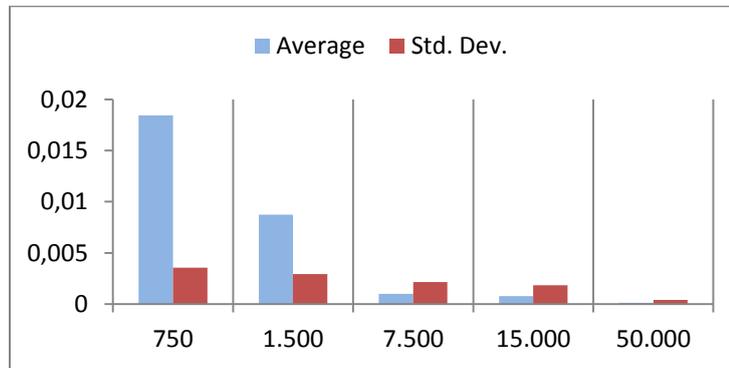


FIGURE 6.5. 3-asset portfolio simulation in the homogeneous profile scenario

Accomplishing a similar procedure with all the other defined portfolios, the results for the mean corresponding to the first 500 iterations are shown in these graphs:



FIGURE 6.6.  $n$ -asset portfolio simulation in the homogeneous profile scenario

The conclusion is similar to the one extracted from the heterogeneous profile scenario, since the convergence is later achieved as the number of assets in the portfolio tends to be higher.

## CONCLUSIONS

These are the main conclusions driven out from this research:

1. Monte Carlo methods result helpful for the valuation of investment portfolio rebalancing strategies, with very promising outcomes.
2. The results of the Monte Carlo simulations converge very rapidly to the exact analytical solution when increasing the number of iterations.
3. The results of the Monte Carlo simulations are consistent when varying the number of assets in the portfolio.
4. The results of the Monte Carlo simulations highlight the existence of the volatility pumping effect, which has already been analytically established.
5. The results of the Monte Carlo simulation demonstrate the existence of the diversification phenomenon, which has already been predicted by the financial theory.

Future research should enlarge the scope of this investigation, by considering the application of the Monte Carlo methods to different investment portfolio rebalancing strategies, with the aim of developing useful tools for academics and financial practitioners.

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