LIMIT BEHAVIOUR OF APPROXIMATE PROPER SOLUTIONS IN VECTOR OPTIMIZATION*

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Abstract. In the framework of a vector optimization problem, we provide conditions for approximate proper solutions to tend to exact weak/efficient/proper solutions when the error tends to zero. This limit behaviour depends on an approximation set that is used to define the approximate proper efficient solutions. We also study the special case when the final space of the vector optimization problem is normed, and more particularly, when it is finite dimensional. In these specific frameworks, we provide several explicit constructions of dilating ordering cones and approximation sets that lead to the desired limit behaviour. In proving our results, new relationships between different concepts of approximate proper efficiency are stated.

Key words. Vector optimization, approximate efficiency, approximate proper efficiency, dilating cone, approximating family of cones

AMS subject classifications. 90C48, 90C26, 90C29

1. Introduction.

When solving a vector optimization problem, heuristic/iterative algorithms are usually employed, specially when the feasible set is too big. However, in practice, when applying these algorithms the accuracy of the solutions is sometimes sacrificed to solve the problem in a reasonable lapse of time. Thus, it is essential to measure the quality of the computed solutions.

With this aim, several notions of approximate efficiency have appeared in the literature. The most known are those ones introduced, respectively, by Kutateladze [17], Németh [19], White [26], Helbig [12] and Tanaka [25]. The common idea in these concepts is to consider a set that approximates the ordering cone, that is, an approximation set similar to the ordering cone, that does not contain the point zero, in order to impose the approximate efficiency (or nondomination) condition in the notions.

This idea motivated the concept of approximate efficiency introduced by Gutiérrez, Jiménez and Novo in [9, 10], in which they considered a general approximation set, in such a way that this concept reduces to the notions defined by the previous authors by taking a specific approximation set for each of them.

On the other hand, the concepts of approximate proper efficiency are more restrictive than the last ones, and arise with the purpose of providing a more depurated approximate efficient set by removing approximate solutions with non desirable properties. The most known are those ones given by Li and Wang [18], Rong [23] and El Maghri [3], in which they combine the approximate efficiency notion due to Kutateladze with, respectively, the proper efficiency concepts introduced by Geoffrion [4],

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Benson [1] and Henig [14].

With the aim of unifying, Gutiérrez, Huerga and Novo [8] and Gutiérrez, Huerga, Jiménez and Novo [7] introduced two notions of approximate proper efficiency based on the more general concept of approximate efficiency stated in [9, 10] and, respectively, the proper efficiency notions by Benson and Henig.

One of the main properties of the approximate proper efficient solutions is that, under generalized convexity conditions, they can be characterized through linear scalarization (see, for instance, [7, 8]), i.e., by means of approximate solutions of scalar optimization problems associated to the original one. This fact is an important advantage in the computation of the solutions, and because of that, the notions of approximate proper efficiency are usually chosen to determine a suitable set of approximate efficient solutions.

Thus, we focus on this type of solutions with the final aim of studying their limit behaviour when the error goes to zero. Depending on the nature of the optimization problem, one may be interested in its exact efficient, weak efficient or proper efficient solution set. Because of that, it is essential to know how to construct an approximation set, that replaces the ordering cone, in such a way that the corresponding approximate proper solutions tend to exact efficient, weak efficient or proper efficient solutions.

In papers [6, 7], a preliminary study of the limit behaviour of approximate proper solutions was made. In both papers, the common purpose was to obtain a sufficient condition for these solutions to tend to exact efficient solutions when the error tends to zero. These sufficient conditions are stronger than the ones presented in this paper and they only focuses on the approximation to the exact efficient set, no results were obtained to approximate neither the weak efficient set nor the proper efficient set.

Furthermore, when the final space of the vector optimization problem is normed, and more particularly, finite dimensional with a polyhedral ordering cone, we provide explicit constructions of the approximation sets, which are easier to handle computationally, overall in the latter setting, in which the approximation sets are defined in terms of matrices.

In this paper we will deal specially with the notion of approximate proper efficiency in the sense of Henig, introduced in [7], and we will determine sufficient conditions that imply the equivalence of these solutions to the approximate proper solutions in the senses of Benson [8] and Geoffrion [18]. These sufficient conditions are essentially based on the existence of a family of dilating cones, that approximate the ordering cone and separate it from another closed cone.

For normed spaces, Sterna-Karwat [24] provided sufficient conditions that guarantee the existence of such a family. Moreover, in the finite dimensional case, Henig [13] proved that one of these families always exists, whenever the ordering cone is closed and pointed. Also, Kaliszewski [16] constructed such a family in the finite dimensional case, when the ordering cone is polyhedral.

The paper is organized as follows. In Section 2 we state the framework, the notations, the main concepts and some previous results. In short Section 3, we study the relationships between the concept of approximate proper efficiency in the sense of Henig, that we use to prove our main results, and some important notions of approximate proper efficiency given in the literature, with the aim of clarifying all the connections among them. Also, we provide equivalent formulations of approximate proper solutions that will be useful for the main Section 4, in which we study the limit behaviour of approximate proper solutions when the precision error goes to zero, and we establish sufficient conditions for approximate proper solutions to tend to an exact weak/efficient/proper solution. We also characterize the set of exact Henig proper efficient solutions through limits of approximate proper efficient solutions, when the error tends to zero, and we particularize these results for the case when the final space is normed, and also when it is finite dimensional with a polyhedral ordering cone, for which more specific and easier constructions of the set of approximate proper solutions are given, thanks to the rich structure of the final space. Finally, in Section 5 we state the conclusions.

2. Preliminaries. Let Y be a real locally convex Hausdorff topological linear space. As usual, we refer to the topological dual space of Y as Y^*. Given a nonempty set $F \subset Y$, we denote by $\operatorname{int} F$, $\operatorname{cl} F$, $\operatorname{bd} F$, F^c, $\operatorname{co} F$ and $\operatorname{cone} F$ the topological interior, the closure, the boundary, the complement, the convex hull and the cone generated by F, respectively. It is said that F is solid if $\operatorname{int} F \neq \emptyset$, and coradiant if $\alpha F \subset F$, for all $\alpha \geq 1$. Moreover, the nonnegative orthant of $\mathbb{R}^r$ is denoted by $\mathbb{R}^r_+$, $\mathbb{R}_+ := \mathbb{R}^1_+$ and we refer to the closed unit ball of a normed space as $\mathcal{B}$.

The polar and strict polar cones of F are denoted by F^+ and F^{s+}, respectively, i.e.,

$$F^+ := \{\lambda \in Y^* : \lambda(y) \geq 0, \forall\, y \in F\},$$
$$F^{s+} := \{\lambda \in Y^* : \lambda(y) > 0, \forall\, y \in F\backslash\{0\}\}.$$

Let $D \subset Y$ be a nonempty convex cone (i.e., $\emptyset \neq D = \mathbb{R}_+ \cdot D = D + D$), which is assumed to be proper ($\{0\} \neq D \neq Y$), closed and pointed ($D \cap (-D) = \{0\}$). From now on, we consider the partial order $\leq_D$ defined on Y by D as usual, i.e.,

$$y_1, y_2 \in Y,\ y_1 \leq_D y_2 \Longleftrightarrow y_2 - y_1 \in D.$$

Next, the notion of approximating family of cones is recalled and some of its main properties and associated concepts are collected (see [2, 13, 20, 21, 24]). It will be a key mathematical tool of this work.

Definition 2.1. *(a) [24, Definition 3.1] Let $\mathcal{F} = \{D_n \subset Y : n \in \mathbb{N}\}$ be a family of decreasing (with respect to the inclusion) solid, closed, pointed convex cones. It is said that $\mathcal{F}$ approximates D if $D\backslash\{0\} \subset \operatorname{int} D_n$ eventually (i.e., there exists $n_0 \in \mathbb{N}$ such that $D\backslash\{0\} \subset \operatorname{int} D_n$, for all $n \geq n_0$) and $D = \bigcap_n D_n$.*

(b) Let $\mathcal{F}$ be an approximating family of cones for D. We say that $\mathcal{F}$ separates D from a closed cone $K \subset Y$ if

$$D \cap K = \{0\} \iff D_n \cap K = \{0\} \text{ eventually.}$$

Remark 2.2. (a) Let $\mathcal{F} = \{D_n\}$ be an approximating family of cones for D that separates D from another closed cone K. If $D \cap K = \{0\}$, then $D\backslash\{0\} \subset \operatorname{int} D_n$ and $K\backslash\{0\} \subset \operatorname{int}(Y\backslash D_n)$ eventually. In other words, D and K are strictly separated by D_n eventually, in the sense of [13, Definition 2.1].

(b) In the finite dimensional setting, each approximating family $\mathcal{F}$ for D fulfills for all fixed $n \in \mathbb{N}$ the stronger inclusion $D_m\backslash\{0\} \subset \operatorname{int} D_n$ eventually (with respect to m), instead of just $D_m \subset D_n$ eventually.

Moreover, if Y is normed, then there exists a family $\mathcal{F}$ approximating D if and only if $D^{s+} \neq \emptyset$ (see [24, Theorem 3.1]).

Observe that condition $D^{s+} \neq \emptyset$ is satisfied if and only if there exists a nonempty closed convex set $B \subset D\backslash\{0\}$ such that $\operatorname{cone} B = D$. This set B is called base of D. For instance, the sets

$$B^\xi := \{d \in D : \xi(d) = 1\}, \quad \forall\, \xi \in D^{s+},$$

are bases of D, and in the finite dimensional setting, they are compact.

In particular, if Y is a separable normed space, we know by the so-called Krein-Rutman theorem (see [15, Theorem 3.38]) that $D^{s+} \neq \emptyset$.

(c) Some authors have explicitly built approximating families of cones in certain settings. For example, Henig [13] obtained an approximating family of cones in the finite dimensional Euclidean space $\mathbb{R}^r$, Kaliszewski [16] for polyhedral cones in finite dimensional spaces, and Sterna-Karwat [24, Theorem 3.1], Borwein and Zhuang [2] and Gong [5] derived this family when Y is normed.

On the other hand, if Y is finite dimensional, then there exist approximating families for D separating from each closed cone K (see [13, Theorem 2.1]), and if Y is normed and D has a weakly compact base, then there exist approximating families for D separating from each weakly closed cone K (see [24, Proposition 6.1]).

For the convenience of the reader next we recall two of these results.

Theorem 2.3. *[24, Proposition 6.1] Let Y be a normed space and suppose that B is a weakly compact base of D. Then the sequence*

$$D_n^B := \operatorname{cone}(B + (1/n)\mathcal{B}), \quad \forall\, n \in \mathbb{N}, \tag{2.1}$$

approximates D and separates it from every weakly closed cone $K \subset Y$.

Consider $Y = \mathbb{R}^r$ and the polyhedral cone

$$P := \{y \in \mathbb{R}^r : Ay \in \mathbb{R}^p_+\}, \tag{2.2}$$

where $A \in \mathcal{M}_{p\times r}$ (i.e., the matrix A has p rows and r columns) and $p \geq r$. In this setting we assume that the elements in $\mathbb{R}^r$ are column vectors. Also, the transpose of a vector $v \in \mathbb{R}^r$ is denoted by v^t. We suppose that $P \neq \{0\}$, which is equivalent to $0 \notin \operatorname{int} \operatorname{co}\{a_i^t : i = 1, 2, \ldots, p\}$, where a_i is the i-th row of A. Moreover, we consider that $\operatorname{rank}(A) = r$. Let us note that P defined in this way is convex, closed and pointed. Moreover, observe that $Ay \in \mathbb{R}^p_+ \backslash \{0\}$ provided that $y \in P\backslash\{0\}$, since P is pointed, and so $u^tAy > 0$ as long as $y \in P\backslash\{0\}$ (i.e., $A^tu \in P^{s+}$), where u is the p-dimensional vector $(1, 1, \ldots, 1)^t$. The following theorem shows a family of cones that approximates P and separates it from every closed cone.

Theorem 2.4. *[16] The sequence*

$$P_n := \{y \in \mathbb{R}^r : Ay + (1/n)uu^tAy \in \mathbb{R}^p_+\}, \quad \forall n \in \mathbb{N}$$

approximates P and separates it from every closed cone $K \subset \mathbb{R}^r$.

Notice that the families $\{D_n^B\}$ and $\{P_n\}$ are strictly decreasing, in the sense that $D_{n+1}^B\backslash\{0\} \subset \operatorname{int} D_n^B$ and $P_{n+1}\backslash\{0\} \subset \operatorname{int} P_n$, for all n. Moreover, for each $n \in \mathbb{N}$, $\zeta := (1/\|A^tu\|)A^tu \in P_n^{+s}$ and

$$B_n^A := \{y \in P_n : \zeta(y) = 1\} \tag{2.3}$$

is a compact base of P_n.

Throughout this paper, we consider the following vector optimization problem:

$$\text{Minimize}_D\ f(x) \text{ subject to } x \in S, \tag{VOP}$$

where $f : X \to Y$, X is an arbitrary decision set and the feasible set $S \subset X$ is nonempty.

Let us recall that a point $x_0 \in S$ is an efficient (resp., weak efficient) solution of problem (VOP), and we denote it by $x_0 \in \mathrm{E}(f, S, D)$ (resp., $x_0 \in \mathrm{WE}(f, S, D)$), if there is not $x \in S$ such that $f(x) \leq_D f(x_0)$, $f(x) \neq f(x_0)$ (resp., $f(x) \leq_{\operatorname{int} D \cup \{0\}} f(x_0)$, $f(x) \neq f(x_0)$). The ordering cone D is assumed to be solid when dealing with weak efficient solutions –otherwise, $\mathrm{WE}(f, S, D) = S$ and weak efficiency is a useless solution concept.

Observe that, for each $x_0 \in S$,

$$x_0 \in \mathrm{E}(f, S, D) \Longleftrightarrow (f(S) - f(x_0)) \cap (-D\backslash\{0\}) = \emptyset,$$
$$x_0 \in \mathrm{WE}(f, S, D) \Longleftrightarrow (f(S) - f(x_0)) \cap (-\operatorname{int} D) = \emptyset.$$

The notions of approximate efficiency that we remind below are defined by following the common idea of replacing the ordering cone by a nonempty set C that approximates it. First, we need to introduce some sets.

For a nonempty set $C \subset Y\backslash\{0\}$, we define the set-valued mapping $C : \mathbb{R}_+ \to 2^Y$ as follows:

$$C(\varepsilon) := \begin{cases} \varepsilon C & \text{if } \varepsilon > 0 \\ (\operatorname{cone} C)\backslash\{0\} & \text{if } \varepsilon = 0, \end{cases}$$

and we introduce the following sets:

$$\mathcal{H} := \{\emptyset \neq C \subset Y\backslash\{0\} : C \cap (-D) = \emptyset\},$$
$$\overline{\mathcal{H}} := \{\emptyset \neq C \subset Y\backslash\{0\} : \operatorname{cl}\operatorname{cone} C \cap (-D) = \{0\}\},$$
$$\mathcal{G}(C) := \left\{ \begin{array}{c} D' \subset Y : D' \text{ is a proper solid convex cone}, \\ D\backslash\{0\} \subset \operatorname{int} D', C \cap (-\operatorname{int} D') = \emptyset \end{array} \right\}.$$

Moreover, given $C \subset Y\backslash\{0\}$, $\varepsilon \geq 0$ and $x \in X$, we denote by $\mathcal{S}(C(\varepsilon), x)$ the set of all families of cones that approximate D and separate D from the cone $-\operatorname{cl}\operatorname{cone}(f(S) + C(\varepsilon) - f(x))$. In particular, condition $\mathcal{S}(C(\varepsilon), x) \neq \emptyset$ means that there exists such a family of cones.

The following approximate efficiency notion due to Gutiérrez, Jiménez and Novo [9] generalizes the most important approximate efficiency concepts defined up to now (see, for instance, [9, 10] and the references therein), which can be recovered by considering specific sets C.

Definition 2.5. *Let $C \in \mathcal{H}$ and $\varepsilon \geq 0$. It is said that $x_0 \in S$ is a (C, ε)-efficient solution of problem* (VOP)*, denoted by $x_0 \in \mathrm{AE}(f, S, C, \varepsilon)$, if*

$$(f(S) - f(x_0)) \cap (-C(\varepsilon)) = \emptyset.$$

Remark 2.6. (a) The (C, ε)-efficiency notion encompasses the concepts of efficient solution and weak efficient solution. To be precise, if $\operatorname{cone} C = D$, then $\mathrm{AE}(f, S, C, 0) = \mathrm{E}(f, S, D)$; if $\operatorname{cone} C = \operatorname{int} D \cup \{0\}$, we have that $\mathrm{AE}(f, S, C, 0) = \mathrm{WE}(f, S, D)$; if $C = D\backslash\{0\}$, then $\mathrm{AE}(f, S, C, \varepsilon) = \mathrm{E}(f, S, D)$, for all $\varepsilon \geq 0$, and if $C = \operatorname{int} D$, then $\mathrm{AE}(f, S, C, \varepsilon) = \mathrm{WE}(f, S, D)$, for all $\varepsilon \geq 0$.

(b) In Definition 2.5 we consider $C \in \mathcal{H}$ to obtain a consistent set of approximate efficient solutions. Indeed, if $C \cap (-D) \neq \emptyset$, it is possible to find simple problems for which the approximate efficient set is empty, for all $\varepsilon > 0$, while the efficient set is not empty (see Remark 2.4 and Example 2.5 in [7]). The following properties hold (see [9, Theorem 3.5(iii)]):

$$\bigcap_{\varepsilon>0} \mathrm{AE}(f,S,C,\varepsilon) = \mathrm{E}(f,S,D), \text{ if } \operatorname{cone} C = D, \tag{2.4}$$

$$\bigcap_{\varepsilon>0} \mathrm{AE}(f,S,C,\varepsilon) = \mathrm{WE}(f,S,D), \text{ if } \operatorname{cone} C = \operatorname{int} D \cup \{0\}. \tag{2.5}$$

With respect to the approximate proper efficiency, the next notion was introduced by Li and Wang in [18] and it extends the concept of proper efficiency in the sense of Geoffrion to the approximate case.

Definition 2.7. *Suppose that* $Y = \mathbb{R}^r$, $D = \mathbb{R}^r_+$ *and let* $\varepsilon \geq 0$ *and* $q \in \mathbb{R}^r_+ \backslash \{0\}$. *A feasible point* x_0 *is a Geoffrion* ε*-proper efficient solution of* (VOP) *with respect to* q, *and it is denoted by* $x_0 \in \mathrm{Ge}(f,S,q,\varepsilon)$, *if there exists* $k > 0$ *such that for each* $x \in S$ *and* $i \in \{1,2,\ldots,r\}$ *with* $f_i(x_0) > f_i(x) + \varepsilon q_i$ *there exists* $j \in \{1,2,\ldots,r\}$ *such that* $f_j(x_0) < f_j(x) + \varepsilon q_j$ *and*

$$\frac{f_i(x_0) - f_i(x) - \varepsilon q_i}{f_j(x) - f_j(x_0) + \varepsilon q_j} \leq k.$$

In particular, if $\varepsilon = 0$ in the above notion, we recover the concept of exact proper efficiency due to Geoffrion [4]. We denote the set of exact proper efficient solutions in the sense of Geoffrion by $\mathrm{Ge}(f,S)$. Notice that $x_0 \in \mathrm{Ge}(f,S,q,\varepsilon)$ if and only if $x_0 \in \mathrm{Ge}(f - \varepsilon q \mathrm{I}_{\{x_0\}}, S)$, where $\mathrm{I}_{\{x_0\}} : X \to \mathbb{R}$ is the indicator function of the singleton $\{x_0\}$.

The next concepts of approximate proper efficiency combine the notions of proper efficiency in the senses of Benson [1] and Henig [14], respectively, with the concept of (C,ε)-efficiency. The first one was introduced by Gutiérrez, Huerga and Novo (see [8]) and the second one by Gutiérrez, Huerga, Jiménez and Novo in [7]. These two notions extend and improve the most important concepts of approximate proper efficiency given in the literature (see, for instance, [7, 8] and the references therein).

Definition 2.8. *Let* $\varepsilon \geq 0$ *and* $C \in \overline{\mathcal{H}}$. *A point* $x_0 \in S$ *is a Benson* (C,ε)*-proper efficient solution of* (VOP), *and we denote it by* $x_0 \in \mathrm{Be}(f,S,C,\varepsilon)$, *if*

$$\operatorname{cl}\operatorname{cone}(f(S) + C(\varepsilon) - f(x_0)) \cap (-D) = \{0\}. \tag{2.6}$$

Definition 2.9. *Let* $\varepsilon \geq 0$ *and* $C \in \overline{\mathcal{H}}$. *A point* $x_0 \in S$ *is a Henig* (C,ε)*-proper efficient solution of* (VOP), *and we denote it by* $x_0 \in \mathrm{He}(f,S,C,\varepsilon)$, *if there exists* $D' \in \mathcal{G}(C)$ *such that* $x_0 \in \mathrm{AE}(f,S,C + \operatorname{int} D', \varepsilon)$.

Remark 2.10. (a) It is clear that $D\backslash\{0\} \in \overline{\mathcal{H}}$, and the concepts of Benson and Henig $(D\backslash\{0\},\varepsilon)$-proper efficiency coincide with the concepts of Benson [1] and Henig [14] proper efficiency, respectively, for all $\varepsilon \geq 0$. Analogously, Benson and Henig $(C,0)$-proper efficiency encompass the concepts of Benson [1] and Henig [14] proper efficiency, respectively, provided that $\operatorname{cl}\operatorname{cone} C = D$. In the sequel, the sets of exact Benson and Henig proper efficient solutions of problem (VOP) are denoted by $\mathrm{Be}(f,S,D)$ and $\mathrm{He}(f,S,D)$, respectively.

(b) The following equivalent formulation for Henig (C,ε)-proper efficient solutions was proved in [7, Theorem 3.3(c)]: A feasible point x_0 is a Henig (C,ε)-proper efficient solution of problem (VOP) if there exists $D' \in \mathcal{G}(C)$, with $\operatorname{int} D' = D'\backslash\{0\}$ such that

$$\operatorname{cl}\operatorname{cone}(f(S) + C(\varepsilon) - f(x_0)) \cap (-\operatorname{int} D') = \emptyset. \tag{2.7}$$

(c) From (2.6) and (2.7) it is easy to see that $\mathrm{He}(f,S,C,\varepsilon) \subset \mathrm{Be}(f,S,C,\varepsilon)$. Moreover, observe that both statements (2.6) and (2.7) imply in particular that $\mathrm{cl}\,\mathrm{cone}\, C \cap (-D) = \{0\}$. Because of that, we consider $C \in \overline{\mathcal{H}}$ in Definitions 2.8 and 2.9.

(d) The concepts of approximate proper efficiency in the senses of Benson and Henig given by the set $C = q + D$, $q \in D\backslash\{0\}$, were introduced, respectively, by Rong [23] and El Maghri [3]. These two concepts and the notion of approximate proper efficiency due to Li and Wang (in the sense of Geoffrion) are based on the notion of approximate efficiency in the sense of Kutateladze [17], in which the approximation error is measured by means of a singleton $\{q\}$.

3. Properties of approximate proper solutions. In this section we state the equivalences between the last concepts of approximate proper efficiency when problem (VOP) is considered, and we establish useful equivalent formulations of the approximate proper solutions in the sense of Henig, that will be needed along the rest of the paper.

THEOREM 3.1. *Let $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. If $\mathcal{S}(C(\varepsilon), x) \neq \emptyset$ for all $x \in S$, then*

$$\mathrm{Be}(f,S,C,\varepsilon) = \mathrm{He}(f,S,C,\varepsilon). \tag{3.1}$$

Proof. Inclusion "$\supset$" in (3.1) is clear from Remark 2.10(c). For proving the other inclusion, let $x_0 \in \mathrm{Be}(f,S,C,\varepsilon)$. By hypothesis we see there exists an approximating family of cones $\{D_n\}$ for D separating from the cone $-\,\mathrm{cl}\,\mathrm{cone}(f(S) + C(\varepsilon) - f(x_0))$, and so

$$\mathrm{cl}\,\mathrm{cone}(f(S) + C(\varepsilon) - f(x_0)) \cap (-D_n) = \{0\} \text{ eventually.}$$

Thus, it follows that $D'_n := \mathrm{int}\, D_n \cup \{0\} \in \mathcal{G}(C)$, $\mathrm{int}\, D'_n = D'_n\backslash\{0\}$, for all n, and they satisfy statement (2.7) eventually, so $x_0 \in \mathrm{He}(f,S,C,\varepsilon)$ by Remark 2.10(b). ◻

In the particular case when Y is finite dimensional, we have the following result.

THEOREM 3.2. *Suppose that $Y = \mathbb{R}^r$ and let $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. Then,*

$$\mathrm{Be}(f,S,C,\varepsilon) = \mathrm{He}(f,S,C,\varepsilon).$$

Moreover, if $D = \mathbb{R}^r_+$ and $q \in \mathbb{R}^r_+\backslash\{0\}$, then

$$\mathrm{Ge}(f,S,q,\varepsilon) = \mathrm{Be}(f,S,q+\mathbb{R}^r_+,\varepsilon) = \mathrm{He}(f,S,q+\mathbb{R}^r_+,\varepsilon). \tag{3.2}$$

Proof. We know that in the finite dimensional setting $Y = \mathbb{R}^r$, there exist approximating families for D separating from each closed cone (see Remark 2.2(c)) and so we only have to prove the first equality in (3.2), since the other ones are clear by Theorem 3.1. Thus, observe from [1, Theorem 3.2] that

$$\begin{aligned} x_0 \in \mathrm{Ge}(f,S,q,\varepsilon) &\iff x_0 \in \mathrm{Ge}(f - \varepsilon q \mathrm{I}_{\{x_0\}}, S) \\ &\iff x_0 \in \mathrm{Be}(f - \varepsilon q \mathrm{I}_{\{x_0\}}, S, \mathbb{R}^r_+). \end{aligned}$$

Furthermore, it is not hard to check that

$$\mathrm{cl}\,\mathrm{cone}((f - \varepsilon q\mathrm{I}_{\{x_0\}})(S) + \mathbb{R}^r_+ - (f - \varepsilon q \mathrm{I}_{\{x_0\}})(x_0)) = \mathrm{cl}\,\mathrm{cone}(f(S) + \mathbb{R}^r_+ - f(x_0) + \varepsilon q)$$

and then the first equality in (3.2) is proved. ◻

Remark 3.3. Let $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. Observe that inclusion "$\supset$" in (3.1) always holds, but inclusion "$\subset$" could be false. However, the equality is satisfied under the assumption $\mathcal{S}(C(\varepsilon), x) \neq \emptyset$ for all $x \in S$. For example, this assumption is true whenever Y is finite dimensional (see [13, Theorem 2.1]); also if Y is normed, D has a weakly compact base and $\operatorname{cl}\operatorname{cone}(f(S)+C(\varepsilon)-f(x))$ is weakly closed for all $x \in S$, as a consequence of Theorem 2.3.

More generally, it is clear from the proof of Theorem 3.1 that only one strict cone separation (see Remark 2.2(a)) is needed. Thus, (3.1) could be also true in some settings different from the setting of Theorem 3.1. For example, [7, Corollary 4.8] states equality (3.1) by supposing that D^+ is solid with respect to a locally convex topology on Y^* compatible with the dual pair (when Y^* is equipped with the topology of uniform convergence on the weakly compact absolutely convex sets of Y, the solidness of D^+ is equivalent to the existence of a weakly compact base of D, see [22]) and $\operatorname{cl}\operatorname{cone}(f(S)+C(\varepsilon)-f(x))$ is convex, for all $x \in S$.

Let us underline that Theorem 3.1 does not require any convexity assumption. From this point of view, it is an improvement of [7, Corollary 4.8]. For instance, in Example 4.12 of this paper, one may deduce by Theorem 3.2 that $\operatorname{He}(f,S,q+P,0.1) = \operatorname{Be}(f,S,q+P,0.1)$ and so $(1.1,1.2) = (1,1)+0.1q \notin \operatorname{Be}(f,S,q+P,0.1)$ (see Figure 1). However, [7, Corollary 4.8] cannot be applied since the set $\operatorname{cl}\operatorname{cone}(f(S)+0.1q+P-f(1.1,1.2))$ is not convex.

The following two theorems will be useful along the paper.

THEOREM 3.4. *Consider* $\varepsilon \geq 0$, $C \in \overline{\mathcal{H}}$, $x_0 \in S$ *and* $\{D_n\} \in \mathcal{S}(C(\varepsilon), x_0)$. *It follows that* $x_0 \in \operatorname{He}(f,S,C,\varepsilon)$ *if and only if* $0 \notin C+G_n$ *and* $x_0 \in \operatorname{AE}(f,S,C+G_n,\varepsilon)$ *eventually, where* $G_n = D_n \backslash\{0\}$ *or* $G_n = \operatorname{int} D_n$, *for all* n.

Proof. Suppose that $x_0 \in \operatorname{He}(f,S,C,\varepsilon)$. Then, by Remark 2.10($c$) we know that $x_0 \in \operatorname{Be}(f,S,C,\varepsilon)$, i.e.,

$$\operatorname{cl}\operatorname{cone}(f(S)-f(x_0)+C(\varepsilon)) \cap (-D) = \{0\}$$

and so

$$\operatorname{cl}\operatorname{cone}(f(S)-f(x_0)+C(\varepsilon)) \cap (-D_n\backslash\{0\}) = \emptyset$$

eventually, since $\{D_n\}$ separates D from $-\operatorname{cl}\operatorname{cone}(f(S)-f(x_0)+C(\varepsilon))$. In particular we have that

$$(f(S)-f(x_0)) \cap (-C(\varepsilon)-D_n\backslash\{0\}) = \emptyset$$

eventually. Thus, $0 \notin C+D_n\backslash\{0\}$ and $x_0 \in \operatorname{AE}(f,S,C+D_n\backslash\{0\},\varepsilon)$ eventually, and so $0 \notin C+\operatorname{int} D_n$ and $x_0 \in \operatorname{AE}(f,S,C+\operatorname{int} D_n,\varepsilon)$ eventually. Notice that $D_n \in \mathcal{G}(C)$ eventually, since $0 \notin C+D_n\backslash\{0\}$ eventually.

The reciprocal implication is clear by the definition. Thus, the proof is finished. □

LEMMA 3.5. *Consider problem* (VOP), $C \subset Y\backslash\{0\}$, $\varepsilon \geq 0$ *and let* $K \subset Y$ *be a solid convex cone such that* $C+K \in \overline{\mathcal{H}}$. *Then,*

$$\operatorname{He}(f,S,C+K,\varepsilon) = \operatorname{He}(f,S,C+(K\backslash\{0\}),\varepsilon) = \operatorname{He}(f,S,C+\operatorname{int} K,\varepsilon).$$

Proof. Let $D' \subset Y$ be an arbitrary solid convex cone. It is not hard to check that

$$K+\operatorname{int} D' = (K\backslash\{0\})+\operatorname{int} D' = \operatorname{int} K+\operatorname{int} D'. \tag{3.3}$$

Therefore, we see that

$$\mathcal{G}(C+K) = \mathcal{G}(C+(K\backslash\{0\})) = \mathcal{G}(C+\operatorname{int} K). \tag{3.4}$$

Moreover, for all $G \in \{K, K\backslash\{0\}, \operatorname{int} K\}$ it is clear that

$$\text{(3.5)} \qquad \operatorname{He}(f, S, C+G, \varepsilon) = \bigcup_{D' \in \mathcal{G}(C+G)} \operatorname{AE}(f, S, C+G+\operatorname{int} D', \varepsilon),$$

and the result follows by (3.3), (3.4) and (3.5). ☐

THEOREM 3.6. *Consider problem* (VOP), $C \subset Y\backslash\{0\}$, $\varepsilon \geq 0$, $x_0 \in S$ *and let* $\{D_n\}$ *be an approximating family of cones for* D *such that* $0 \notin C + D_{\bar{n}}$ *for some* $\bar{n}$. *Suppose that* $\{D_n\} \in \mathcal{S}((C+D_{\bar{n}})(\varepsilon), x_0)$. *Then, for each* $G_{\bar{n}} \in \{D_{\bar{n}}, D_{\bar{n}}\backslash\{0\}, \operatorname{int} D_{\bar{n}}\}$,

$$\text{(3.6)} \qquad x_0 \in \operatorname{He}(f, S, C+G_{\bar{n}}, \varepsilon) \iff x_0 \in \operatorname{AE}(f, S, C+\operatorname{int} D_{\bar{n}}, \varepsilon).$$

Proof. First, observe that $C + D_{\bar{n}} \in \overline{\mathcal{H}}$ since $0 \notin C + D_{\bar{n}}$. Then, $C + G_{\bar{n}} \in \overline{\mathcal{H}}$, for all $G_{\bar{n}} \in \{D_{\bar{n}}, D_{\bar{n}}\backslash\{0\}, \operatorname{int} D_{\bar{n}}\}$. By Lemma 3.5 we see that

$$\operatorname{He}(f, S, C+D_{\bar{n}}, \varepsilon) = \operatorname{He}(f, S, C+(D_{\bar{n}}\backslash\{0\}), \varepsilon) = \operatorname{He}(f, S, C+\operatorname{int} D_{\bar{n}}, \varepsilon).$$

Then the result follows by proving statement (3.6) for $G_{\bar{n}} = D_{\bar{n}}$.

Let $x_0 \in \operatorname{He}(f, S, C+D_{\bar{n}}, \varepsilon)$. By applying Theorem 3.4 we deduce that $0 \notin C+D_{\bar{n}}+\operatorname{int} D_n$ and $x_0 \in \operatorname{AE}(f, S, C+D_{\bar{n}}+\operatorname{int} D_n, \varepsilon)$ eventually. Consider an arbitrary $n' \in \mathbb{N}$, $n' > \bar{n}$, such that $x_0 \in \operatorname{AE}(f, S, C+D_{\bar{n}}+\operatorname{int} D_{n'}, \varepsilon)$. As the family $\{D_n\}$ is decreasing we have that $D_{\bar{n}} + \operatorname{int} D_{n'} = \operatorname{int} D_{\bar{n}}$ and so $x_0 \in \operatorname{AE}(f, S, C+\operatorname{int} D_{\bar{n}}, \varepsilon)$.

The reciprocal implication is a direct consequence of the definition and the proof finishes. ☐

From Theorems 3.4 and 3.6 we obtain the next corollary.

COROLLARY 3.7. *Consider problem* (VOP), $C \in \overline{\mathcal{H}}$, $\varepsilon \geq 0$ *and*

$$\{D_n\} \in \bigcap_{x \in S} \mathcal{S}(C(\varepsilon), x)$$

such that for each $x \in S$, $\{D_n\} \in \mathcal{S}((C+D_m)(\varepsilon), x)$ *eventually. It follows that*

$$\begin{aligned}
\operatorname{He}(f, S, C, \varepsilon) &= \bigcup_{\{n: 0 \notin C+D_n\}} \operatorname{AE}(f, S, C+(D_n\backslash\{0\}), \varepsilon) \\
&= \bigcup_{\{n: 0 \notin C+\operatorname{int} D_n\}} \operatorname{AE}(f, S, C+\operatorname{int} D_n, \varepsilon) \\
&= \bigcup_{\{n: 0 \notin C+D_n\}} \operatorname{He}(f, S, C+G_n, \varepsilon),\ \forall\, G_n \in \{D_n, D_n\backslash\{0\}, \operatorname{int} D_n\}.
\end{aligned}$$

The exact version of Corollary 3.7 is stated in the next result, which is deduced by considering $C = D\backslash\{0\}$ and $\varepsilon = 1$.

COROLLARY 3.8. *Consider problem* (VOP) *and* $\{D_n\} \in \bigcap_{x \in S} \mathcal{S}(D, x)$ *such that for each* $x \in S$, $\{D_n\} \in \mathcal{S}(D_m, x)$ *eventually. It follows that*

$$\operatorname{He}(f, S, D) = \bigcup_n \operatorname{WE}(f, S, D_n).$$

If additionally, for each n *we have* $D_m\backslash\{0\} \subset \operatorname{int} D_n$ *eventually, then*

$$\operatorname{He}(f, S, D) = \bigcup_n \operatorname{WE}(f, S, D_n) = \bigcup_n \operatorname{E}(f, S, D_n) = \bigcup_n \operatorname{He}(f, S, D_n).$$

In the finite dimensional case, we have the following result.

COROLLARY 3.9. *Consider problem* (VOP) *and suppose that* $Y = \mathbb{R}^r$.
(a) For each compact base B *of* D *it follows that*

$$\mathrm{He}(f,S,D) = \bigcup_n \mathrm{WE}(f,S,D_n^B) = \bigcup_n \mathrm{E}(f,S,D_n^B) = \bigcup_n \mathrm{He}(f,S,D_n^B).$$

(b) If $D = P$, *where* P *is the polyhedral cone defined in* (2.2), *then*

$$\mathrm{He}(f,S,P) = \bigcup_n \mathrm{WE}(f,S,P_n) = \bigcup_n \mathrm{E}(f,S,P_n) = \bigcup_n \mathrm{He}(f,S,P_n).$$

4. Limit behaviour. In this section we are going to study the limit behaviour of Henig (C,ε)-proper efficient solutions of (VOP), when ε tends to zero, for specific sets $C \in \overline{\mathcal{H}}$.

As we will see below, depending on the selected set, it is possible to reach exact weak/efficient/proper solutions in terms of limits of sequences of Henig (C,ε)-proper efficient solutions of (VOP), when ε tends to zero.

The selection of C to compute a suitable approximation of the efficient/weak efficient/proper efficient set is relevant, as it is shown in the following illustrative example.

Example 4.1. Let $X = Y = \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity function on $\mathbb{R}^2$, and $S = D = \mathbb{R}^2_+$. It is clear that $\mathrm{E}(f,S,D) = \{(0,0)^t\}$. Let $\varepsilon > 0$ and $q = (1,1)^t \in \mathbb{R}^2_+$. Then, it is easy to check that

$$\begin{aligned} \mathrm{AE}(f,S,q+\mathbb{R}^2_+,\varepsilon) &= \mathbb{R}^2_+ \cap ((\varepsilon,\varepsilon)^t + \mathbb{R}^2_+)^c, \\ \mathrm{He}(f,S,q+\mathbb{R}^2_+,\varepsilon) &= \mathrm{AE}(f,S,q+\mathbb{R}^2_+,\varepsilon) \cup \{(\varepsilon,\varepsilon)^t\}. \end{aligned}$$

Thus, for any $\varepsilon > 0$ these sets of approximate solutions do not provide good approximations of the efficient set. In fact, what they provide is a suitable approximation of the weak efficient set.

On the other hand, if we now consider $C = \mathrm{co}\{(1,0)^t,(0,1)^t\} + D$, then one can easily see that $\mathrm{AE}(f,S,C,\varepsilon) = \{(x_1,x_2)^t \in \mathbb{R}^2_+ : x_2 < \varepsilon - x_1\}$. In this case, the set of approximate solutions is bounded and for $\varepsilon > 0$ small enough it represents a good approximation of the efficient set.

In the next theorem, we characterize the set of exact efficient and proper efficient solutions of (VOP) as intersections of sets of approximate proper efficient solutions. A previous lemma is needed.

LEMMA 4.2. *Let* $B \subset Y$ *be a base of* D. *Then,*

$$\delta B + (D\backslash\{0\}) = \bigcup_{\varepsilon>\delta} \varepsilon B + (D\backslash\{0\}), \quad \forall \delta \geq 0.$$

Proof. Let $\delta \geq 0$ and $\varepsilon > \delta$. As $B \subset D\backslash\{0\}$, it is clear that

$$\begin{aligned} \varepsilon B + (D\backslash\{0\}) &\subset \delta B + (\varepsilon-\delta)B + (D\backslash\{0\}) \subset \delta B + (D\backslash\{0\}) + (D\backslash\{0\}) \\ &= \delta B + (D\backslash\{0\}). \end{aligned}$$

Reciprocally, let $b \in B$ and $d \in D\backslash\{0\}$ arbitrary. There exists $\lambda > 0$ and $b' \in B$ such that $d = \lambda b'$. Thus,

$$\delta b + d = (\delta+\lambda)\left(\frac{\delta}{\delta+\lambda} b + \frac{\lambda}{\delta+\lambda} b'\right).$$

We have that $b'' := (\delta/(\delta+\lambda))b + (\lambda/(\delta+\lambda))b' \in B$, since B is convex. Therefore,

$$\delta b + d = (\delta+\lambda)b'' = (\delta+\lambda/2)b'' + (\lambda/2)b'' \in \bigcup_{\varepsilon>\delta} \varepsilon B + (D\backslash\{0\}),$$

which finishes the proof. ∎

Theorem 4.3. *Let $B \subset Y$ be a base of D. The following statements hold.*

(a) $\mathrm{He}(f,S,D) \subset \bigcap_{\varepsilon>0} \mathrm{He}(f,S,B+D,\varepsilon) \subset \bigcap_{\varepsilon>0} \mathrm{He}(f,S,q+D,\varepsilon) \subset \mathrm{WE}(f,S,D),$ *for any $q \in D\backslash\{0\}$.*

(b) $\bigcap_{\varepsilon>\delta} \mathrm{He}(f,S,B+D,\varepsilon) \subset \mathrm{AE}(f,S,B+(D\backslash\{0\}),\delta)$, *for all $\delta \geq 0$.*

(c) Suppose that B is weakly compact, there exists an approximating family for D, $f(S) = Q+H$, Q is a weakly compact set of Y and $H \subset D$, $0 \in H$. Then,

$$\mathrm{AE}(f,S,B+D,\varepsilon) \subset \mathrm{He}(f,S,B+D,\varepsilon), \quad \forall \varepsilon > 0.$$

(d) Under the assumptions of part (c), it follows that

$$\bigcap_{\varepsilon>0} \mathrm{AE}(f,S,B+D,\varepsilon) = \bigcap_{\varepsilon>0} \mathrm{He}(f,S,B+D,\varepsilon) = \mathrm{E}(f,S,D).$$

(e) Assume that $\{D_n\} \in \bigcap_{x\in S} \mathcal{S}(D,x)$ and consider a sequence $\{C_n\}$ of sets in Y such that $C_n \subset D_n\backslash\{0\}$ and $\operatorname{int} D_n \subset (C_n + D\backslash\{0\})(0)$, for all $n \in \mathbb{N}$. Then,

$$\bigcup_n \bigcap_{\varepsilon>0} \mathrm{He}(f,S,C_n,\varepsilon) = \mathrm{He}(f,S,D).$$

Proof. (a) The first inclusion is a particular case of [7, Theorem 3.6(f)], since $B+D \subset D\backslash\{0\}$, and the third inclusion is a direct consequence of [7, Remark 3.2(d)] and [10, Theorem 3.4(iii)], since

$$\operatorname{int} D \subset \operatorname{cone}(q + D\backslash\{0\})\backslash\{0\} = (q + D\backslash\{0\})(0) \quad \forall q \in D\backslash\{0\}.$$

For deriving the second inclusion, note that for every $q \in D\backslash\{0\}$ there exists $\lambda > 0$ and $b \in B$ such that $q = \lambda b$, so $(1/\lambda)q \in B$. Then, by [7, Theorem 3.6(b)] $\mathrm{He}(f,S,B+D,\varepsilon) \subset \mathrm{He}(f,S,q+D,\varepsilon/\lambda)$, for all $\varepsilon > 0$, and we have that

$$\bigcap_{\varepsilon>0} \mathrm{He}(f,S,B+D,\varepsilon) \subset \bigcap_{\varepsilon>0} \mathrm{He}(f,S,q+D,\varepsilon).$$

(b) Let $\delta \geq 0$. By [7, Remark 3.2(d)] it is clear that

$$\bigcap_{\varepsilon>\delta} \mathrm{He}(f,S,B+D,\varepsilon) \subset \bigcap_{\varepsilon>\delta} \mathrm{AE}(f,S,B+(D\backslash\{0\}),\varepsilon).$$

and then the result follows by Lemma 4.2.

(c) Consider $\varepsilon > 0$ and $x_0 \in \mathrm{AE}(f,S,B+D,\varepsilon)$. By the assumptions we deduce that $H + D = D$ and then

$$(Q - f(x_0)) \cap (-\varepsilon B - D) = \emptyset. \tag{4.1}$$

Reasoning by contradiction suppose that $x_0 \notin \mathrm{He}(f,S,B+D,\varepsilon)$ and let $\{D_n\}$ be an approximating family for D. Then, $x_0 \notin \mathrm{AE}(f,S,B+D+\operatorname{int} D_n,\varepsilon)$, for all $n \in \mathbb{N}$.

As for each n, $H + D + \operatorname{int} D_n = \operatorname{int} D_n$, through the same reasoning as before we deduce that

$$(Q - f(x_0)) \cap (-\varepsilon B - \operatorname{int} D_n) \neq \emptyset \quad \forall\, n \in \mathbb{N}.$$

Then there exist sequences $(q_n) \subset Q$, $(b_n) \subset B$ and $(d_n) \subset Y$ such that $d_n \in \operatorname{int} D_n$ and $q_n - f(x_0) = -\varepsilon b_n - d_n$, for all n. By compactness, taking subsequences if necessary, we can assume that $q_n \overset{w}{\to} q \in Q$, $b_n \overset{w}{\to} b \in B$, so $d_n \overset{w}{\to} -q + f(x_0) - \varepsilon b$ and by the definition of approximating family of cones it follows that $-q + f(x_0) - \varepsilon b \in D$. Thus, $(Q - f(x_0)) \cap (-\varepsilon B - D) \neq \emptyset$ and we reach a contradiction with statement (4.1).

(d) It follows by (2.4) and as a direct consequence of parts (b) and (c), since

$$\begin{aligned} \mathrm{E}(f, S, D) &= \bigcap_{\varepsilon>0} \mathrm{AE}(f, S, B + D, \varepsilon) \subset \bigcap_{\varepsilon>0} \mathrm{He}(f, S, B + D, \varepsilon) \\ &\subset \mathrm{AE}(f, S, B + (D\backslash\{0\}), 0) = \mathrm{E}(f, S, D). \end{aligned}$$

(e) First, let us observe that, for each $n \in \mathbb{N}$, condition $C_n \subset D_n\backslash\{0\}$ implies $C_n \in \overline{\mathcal{H}}$ and

$$(C_n + \operatorname{int} D_n)(0) = \operatorname{int} D_n. \tag{4.2}$$

Let $x_0 \in \mathrm{He}(f, S, D)$. By applying Theorem 3.4 with $C = D\backslash\{0\}$ and $\varepsilon = 1$ we deduce that $x_0 \in \mathrm{AE}(f, S, \operatorname{int} D_n, 1)$ eventually. Thus, there exists $m \in \mathbb{N}$ such that

$$x_0 \in \mathrm{AE}(f, S, \operatorname{int} D_m, 1) = \mathrm{WE}(f, S, D_m) = \bigcap_{\varepsilon>0} \mathrm{AE}(f, S, C_m + \operatorname{int} D_m, \varepsilon),$$

where the last equality is a consequence of (4.2) and (2.5).

It is clear by Definition 2.9 that

$$\mathrm{AE}(f, S, C_m + \operatorname{int} D_m, \varepsilon) \subset \mathrm{He}(f, S, C_m, \varepsilon), \quad \forall \varepsilon > 0$$

and so we have that

$$x_0 \in \bigcup_n \bigcap_{\varepsilon>0} \mathrm{He}(f, S, C_n, \varepsilon).$$

Reciprocally, for each $n \in \mathbb{N}$, by [7, Remark 3.2(d)], [10, Theorem 3.4(iii)] and assumption $\operatorname{int} D_n \subset (C_n + D\backslash\{0\})(0)$ we see that

$$\begin{aligned} \bigcup_n \bigcap_{\varepsilon>0} \mathrm{He}(f, S, C_n, \varepsilon) &\subset \bigcup_n \bigcap_{\varepsilon>0} \mathrm{AE}(f, S, C_n + D\backslash\{0\}, \varepsilon) \\ &= \bigcup_n \mathrm{AE}(f, S, C_n + D\backslash\{0\}, 0) \\ &\subset \bigcup_n \mathrm{WE}(f, S, D_n) \\ &\subset \mathrm{He}(f, S, D) \end{aligned}$$

and the proof finishes. □

Remark 4.4. (a) Condition $C_n \subset D_n\backslash\{0\}$ is equivalent to the following one:

$$0 \notin C_n \text{ and } C_n + D\backslash\{0\} \subset \operatorname{int} D_n.$$

Thus, the assumptions on the sets C_n in Theorem 4.3(e) can be reformulated as follows: $0 \notin C_n$ and $(C_n + D\backslash\{0\})(0) = \operatorname{int} D_n$, for all n. For instance, this condition is satisfied by $C_n \in \{G_n + D_n, B_n + D\}$, where $G_n \subset D_n\backslash\{0\}$ and B_n is a base of D_n. A very easy family to construct satisfying the last condition is $\{q + D_n\}$, for $q \in D\backslash\{0\}$.

(b) Let $B \subset Y$ be a base of D. By [7, Theorem 3.6(b)] we have that

$$\bigcap_{\varepsilon \geq \delta} \mathrm{He}(f, S, B + D, \varepsilon) = \mathrm{He}(f, S, B + D, \delta), \quad \forall \delta \geq 0$$

and by applying parts (a) and (b) of Theorem 4.3 we deduce that

$$\mathrm{He}(f, S, D) \subset \bigcap_{\varepsilon > 0} \mathrm{He}(f, S, B + D, \varepsilon) \subset \mathrm{E}(f, S, D).$$

If additionally the assumptions of part (c) are fulfilled, by part (d) we know that

$$\bigcap_{\varepsilon > 0} \mathrm{He}(f, S, B + D, \varepsilon) = \mathrm{E}(f, S, D) \tag{4.3}$$

and also

$$\begin{aligned} \mathrm{AE}(f, S, B + D, \delta) \subset \mathrm{He}(f, S, B + D, \delta) &\subset \bigcap_{\varepsilon > \delta} \mathrm{He}(f, S, B + D, \delta) \\ &\subset \mathrm{AE}(f, S, B + (D\backslash\{0\}), \delta), \quad \forall \delta > 0. \end{aligned}$$

Under the assumptions of Theorem 4.3(c), we deduce from (4.3) that for $\delta > 0$ small enough the set $\bigcap_{\varepsilon \geq \delta} \mathrm{He}(f, S, B + D, \varepsilon) = \mathrm{He}(f, S, B + D, \delta)$ is a good approximation of the efficient set, and two proper estimations for $\mathrm{He}(f, S, B + D, \delta)$ are the sets $\mathrm{AE}(f, S, B + D, \delta)$ and $\mathrm{AE}(f, S, B + D\backslash\{0\}, \delta)$. In particular, it must be underlined that the set of Henig $(B + D, \delta)$-proper efficient solutions represents suitably the efficient set.

On the other hand, notice by the proof of Theorem 4.3(e) that, for each $x_0 \in \mathrm{He}(f, S, D)$ it follows that $x_0 \in \bigcap_{\varepsilon > 0} \mathrm{He}(f, S, C_n, \varepsilon)$ eventually. Then, for $n \in \mathbb{N}$ big enough, the set $\bigcap_{\varepsilon > 0} \mathrm{He}(f, S, C_n, \varepsilon)$ may be a good approximation of the set $\mathrm{He}(f, S, D)$. As $\bigcap_{\varepsilon \geq \delta} \mathrm{He}(f, S, C_n, \varepsilon)$ approximates the set $\bigcap_{\varepsilon > 0} \mathrm{He}(f, S, C_n, \varepsilon)$ for $\delta > 0$ small enough, then it also approximates suitably the set of exact Henig proper solutions of problem (VOP).

Moreover, we can simplify expression $\bigcap_{\varepsilon \geq \delta} \mathrm{He}(f, S, C_n, \varepsilon)$ by considering approximation sets that satisfy certain properties. For example, if C_n are coradiant sets, then [7, Theorem 3.6(c)] can be applied and then

$$\bigcap_{\varepsilon \geq \delta} \mathrm{He}(f, S, C_n, \varepsilon) = \mathrm{He}(f, S, C_n, \delta).$$

In the following two theorems, we establish sufficient conditions for exact Henig proper efficient, efficient and weak efficient solutions in terms of limits of sequences of Henig approximate proper efficient solutions of (VOP).

Previously, a lemma is needed in order to derive part (c) of Theorem 4.6. It extends [7, Lemma 3.7] to any (not necessarily finite dimensional) linear space Y and any base B of the ordering cone D.

Lemma 4.5. *Let $B \subset Y$ be a base of D and consider two sequences $(\varepsilon_k) \subset \mathbb{R}_+\backslash\{0\}$ and $(y_k) \subset Y$, and a point $y \in Y$ such that $\varepsilon_k \to 0$, $y_k \to y$, $y_{k+1} \leq_D y_k$ and*

$$y_k \in D \cap (Y\backslash (\varepsilon_k B + (D\backslash\{0\}))), \quad \forall k \in \mathbb{N}.$$

Then, $y = 0$.

Proof. As D is closed we have that $y \in D$. Moreover, since $y_{k+1} \leq_D y_k$ for all k, it is easy to check that $y \leq_D y_k$, for all k.

Suppose, reasoning by contradiction, that $y \neq 0$. Then, by Lemma 4.2 with $\delta = 0$ there exists $\overline{\varepsilon} > 0$ such that $y \in \overline{\varepsilon}B + (D\backslash\{0\})$ and for each $k \in \mathbb{N}$ such that $\varepsilon_k \leq \overline{\varepsilon}$ we obtain that $y \in \varepsilon_k B + (D\backslash\{0\})$. Fix $k_0 \in \mathbb{N}$ such that $\varepsilon_{k_0} \leq \overline{\varepsilon}$. Then,

$$y_{k_0} = y + (y_{k_0} - y) \in \varepsilon_{k_0} B + (D\backslash\{0\}) + D = \varepsilon_{k_0} B + (D\backslash\{0\}),$$

which is a contradiction. Therefore, $y = 0$ and the proof finishes. □

Theorem 4.6. *In problem (VOP) consider $C \in \overline{\mathcal{H}}$, $x_0 \in S$ and sequences $(x_k) \subset X$ and $(\varepsilon_k) \subset \mathbb{R}_+\backslash\{0\}$ such that $x_k \in \mathrm{He}(f, S, C, \varepsilon_k)$, for all $k \in \mathbb{N}$, $\varepsilon_k \downarrow 0$ and $f(x_k) \to f(x_0)$.*

(a) If $C = G + K$, where $K \in \mathcal{G}(D\backslash\{0\})$, $G \subset K\backslash(-K)$, then $x_0 \in \mathrm{He}(f, S, D)$.

(b) If D is solid and $C = G + D$, where $G \subset D\backslash\{0\}$, then $x_0 \in \mathrm{WE}(f, S, D)$.

(c) If B is a base of D, $C = B + D$ and $f(x_{k+1}) \leq_D f(x_k)$, for all k, then $x_0 \in \mathrm{E}(f, S, D)$.

Proof. (a) First, observe that $G + K \in \overline{\mathcal{H}}$. By [7, Remark 3.2(d)] we see that

$$x_k \in \mathrm{AE}(f, S, G + K + D\backslash\{0\}, \varepsilon_k), \quad \forall\, k.$$

We have that $K + D\backslash\{0\} = \operatorname{int} K$, since $K \in \mathcal{G}(D\backslash\{0\})$. Moreover, $G + \operatorname{int} K$ is coradiant. Then, by [10, Theorem 3.4(iv)] we deduce

$$x_0 \in \mathrm{AE}(f, S, G + \operatorname{int} K, 0) = \mathrm{WE}(f, S, K),$$

since $(G + \operatorname{int} K)(0) = \operatorname{int} K$, and the result follows since $\mathrm{WE}(f, S, K) \subset \mathrm{He}(f, S, D)$.

(b) By [7, Remark 3.2(d)] we deduce that

$$x_k \in \mathrm{AE}(f, S, G + D + D\backslash\{0\}, \varepsilon_k) \subset \mathrm{AE}(f, S, G + \operatorname{int} D, \varepsilon_k), \quad \forall\, k \in \mathbb{N},$$

since $D + D\backslash\{0\} = D\backslash\{0\} \supset \operatorname{int} D$. From here, by reasoning in analogous way as in part (a), we conclude that $x_0 \in \mathrm{WE}(f, S, D)$.

(c) This result follows by applying [11, Corollary 7(b)] to the data $K = D$, $M = f(S)$ and $G(\varepsilon) = \varepsilon B + D\backslash\{0\}$ (Lemma 4.5 ensures that the assumptions of [11, Corollary 7(b)] are fulfilled). □

Remark 4.7. (a) If $\mathcal{S}(C(\varepsilon_k), x) \neq \emptyset$, for all k and for all $x \in S$, then by Theorem 3.1 the approximate Benson and Henig proper solution sets coincide, and we have that the accuracy of Theorem 4.6(a) is better than in [6, Theorem 3.7 c)], since in Theorem 4.6(a) it is proved that the approximate proper solutions tend to exact efficient solutions which are proper solutions.

(b) Part (c) of Theorem 4.6 extends [7, Theorem 3.8] to any (not necessarily finite dimensional) linear space Y and any base B of the ordering cone D.

(c) The easiest way to apply the previous theorem is by considering a singleton $G = \{q\}$, where $q \in K\backslash(-K)$ in part (a) and $q \in D\backslash\{0\}$ in part (b).

In the particular case when Y is normed or finite dimensional, we obtain the following results as consequences of Theorem 4.6.

For the next result, we suppose that Y is normed and we consider the family of cones $\{D_n^B\}$ introduced in (2.1), for a base B of D. We denote $D_\infty^B = D$ and $B_\infty := B$.

Let us also denote by $\bar{n}$ a natural number big enough so that $0 \notin B_n := B+(1/n)\mathcal{B}$. We have that $B_n + D_m^B \in \overline{\mathcal{H}}$, $\forall n, m \in \mathbb{N} \cup \{\infty\}$, $n, m \geq \bar{n}$.

COROLLARY 4.8. *Suppose that Y is normed and $B \subset D\backslash\{0\}$ is a base of D. Let $x_0 \in S$, $n_1, n_2 \in \mathbb{N} \cup \{\infty\}$, $n_1, n_2 \geq \bar{n}$ and let $(x_k) \subset X$ and $(\varepsilon_k) \subset \mathbb{R}_+\backslash\{0\}$ be two sequences such that $x_k \in \mathrm{He}(f, S, B_{n_1} + D_{n_2}^B, \varepsilon_k)$, for all $k \in \mathbb{N}$, $\varepsilon_k \downarrow 0$ and $f(x_k) \to f(x_0)$.*

(a) If $n_2 \neq \infty$, then $x_0 \in \mathrm{He}(f, S, D)$.

(b) If D is solid, then $x_0 \in \mathrm{WE}(f, S, D)$.

(c) If $f(x_{k+1}) \leq_D f(x_k)$, for all $k \in \mathbb{N}$, then $x_0 \in \mathrm{E}(f, S, D)$.

Proof. (i) As $B + D_{n_2}^B \subset B_{n_1} + D_{n_2}^B$, by [7, Theorem 3.6$(b)$] we have that

$$\mathrm{He}(f, S, B_{n_1} + D_{n_2}^B, \varepsilon_k) \subset \mathrm{He}(f, S, B + D_{n_2}^B, \varepsilon_k).$$

Then by applying Theorem 4.6(a) with $G = B$ and $K = D_{n_2}^B$ we see that $x_0 \in \mathrm{He}(f, S, D)$. For parts (b) and (c) observe that since $B + D \subset B_{n_1} + D_{n_2}^B$, by [7, Theorem 3.6$(b)$] we have that

$$\mathrm{He}(f, S, B_{n_1} + D_{n_2}^B, \varepsilon_k) \subset \mathrm{He}(f, S, B + D, \varepsilon_k).$$

Thus, if D is solid, Theorem 4.6(b) implies that $x_0 \in \mathrm{WE}(f, S, D)$ and if $f(x_{k+1}) \leq_D f(x_k)$ for all $k \in \mathbb{N}$, by applying Theorem 4.6(c) we see that $x_0 \in \mathrm{E}(f, S, D)$. ∎

In the next corollary, we consider that $Y = \mathbb{R}^r$ and D is the polyhedral cone P defined in (2.2). We are going to work with the approximating family of cones $\{P_n\}$ stated in Theorem 2.4.

For each n, we remind that B_n^A is the base of P_n defined in (2.3). Denote $P_\infty = P$ and $B_\infty^A := \{y \in P : \zeta(y) = 1\}$.

The proof of this corollary follows from Theorem 4.6, reasoning in analogous way as in the corollary above.

COROLLARY 4.9. *Suppose that $Y = \mathbb{R}^r$. Let $x_0 \in S$, $n_1, n_2 \in \mathbb{N} \cup \{\infty\}$ and let $(x_k) \subset X$ and $(\varepsilon_k) \subset \mathbb{R}_+\backslash\{0\}$ be two sequences such that $x_k \in \mathrm{He}(f, S, B_{n_1}^A + P_{n_2}, \varepsilon_k)$, for all $k \in \mathbb{N}$, $\varepsilon_k \downarrow 0$ and $f(x_k) \to f(x_0)$.*

(a) If $n_2 \neq \infty$, then $x_0 \in \mathrm{He}(f, S, P)$.

(b) If P is solid, then $x_0 \in \mathrm{WE}(f, S, P)$.

(c) If $f(x_{k+1}) \leq_P f(x_k)$ for all $k \in \mathbb{N}$, then $x_0 \in \mathrm{E}(f, S, P)$.

Let X be a Hausdorff topological space and let $F : \mathbb{R}_+ \to 2^X$ be a set-valued mapping. We remind that $x_0 \in X$ belongs to the upper limit of F when $\varepsilon \to 0$, and we denote it by $x_0 \in \limsup_{\varepsilon\to 0} F(\varepsilon)$, if there exist sequences $(\varepsilon_k) \subset \mathbb{R}_+\backslash\{0\}$, $\varepsilon_k \to 0$ and $(x_k) \subset X$, such that $x_k \in F(\varepsilon_k)$, for all $k \in \mathbb{N}$ and $x_k \to x_0$.

In the next theorem we formulate the exact proper and weak efficient solutions of (VOP) in terms of the upper limit of approximate proper solutions when ε tends to zero.

THEOREM 4.10. *Consider problem (VOP) and assume that X is a Hausdorff topological space, f is continuous on S and S is closed.*

(a) Let $\{D_n\} \in \bigcap_{x\in S} \mathcal{S}(D,x)$ *and* $\{G_n\}$ *be a sequence of nonempty sets in* Y *such that* $G_n \subset D_n\backslash\{0\}$, *for all* $n \in \mathbb{N}$. *Then*

$$\bigcup_n \limsup_{\varepsilon\to 0} \mathrm{He}(f,S,G_n+D_n,\varepsilon) = \mathrm{He}(f,S,D).$$

(b) If D *is solid and* $G \subset D\backslash\{0\}$, *then*

$$\limsup_{\varepsilon\to 0} \mathrm{He}(f,S,G+D,\varepsilon) \subset \limsup_{\varepsilon\to 0} \mathrm{AE}(f,S,G+\operatorname{int} D,\varepsilon) = \mathrm{WE}(f,S,D).$$

Proof. (a) Let $n \in \mathbb{N}$ arbitrary. The inclusion

$$\limsup_{\varepsilon\to 0} \mathrm{He}(f,S,G_n+D_n,\varepsilon) \subset \mathrm{He}(f,S,D)$$

follows directly by applying Theorem 4.6(a) to the sets $G = G_n$, $K = D_n$ and taking into account that f is continuous on S and S is closed.

Reciprocally, let $x_0 \in \mathrm{He}(f,S,D)$. By considering $C_n = G_n + D_n$ in Theorem 4.3(e) we obtain that there exists $m \in \mathbb{N}$ such that $x_0 \in \bigcap_{\varepsilon>0} \mathrm{He}(f,S,G_m+D_m,\varepsilon)$ and part (a) is proved.

(b) By [7, Remark 3.2(d)] we deduce the inclusion

$$\limsup_{\varepsilon\to 0} \mathrm{He}(f,S,G+D,\varepsilon) \subset \limsup_{\varepsilon\to 0} \mathrm{AE}(f,S,G+\operatorname{int} D,\varepsilon).$$

On the other hand, it is not hard to check that the sets $\mathrm{AE}(f,S,G+\operatorname{int} D,\varepsilon)$ are closed. Moreover, since $G+\operatorname{int} D$ is coradiant, by [10, Theorem 3.4(ii)] the collection of these sets is decreasing with respect to $\varepsilon > 0$. Thus,

$$\limsup_{\varepsilon\to 0} \mathrm{AE}(f,S,G+\operatorname{int} D,\varepsilon) = \bigcap_{\varepsilon>0} \mathrm{AE}(f,S,G+\operatorname{int} D,\varepsilon) = \mathrm{WE}(f,S,D)$$

where the last equality is obtained by taking into account that $(G+\operatorname{int} D)(0) = \operatorname{int} D$ and statement (2.5), and the proof is finished. □

Remark 4.11. (a) As in Theorem 4.6, the more effective way to apply parts (a) and (b) of Theorem 4.10 is consider in part (a) singletons $G_n = \{q_n\}$, where $q_n \in D_n\backslash\{0\}$ for all $n \in \mathbb{N}$, and $G = \{q\}$ with $q \in D\backslash\{0\}$ in part (b).

(b) In Theorem 4.10(a), inclusion

$$\bigcup_n \limsup_{\varepsilon\to 0} \mathrm{He}(f,S,G_n+D_n,\varepsilon) \subset \mathrm{He}(f,S,D) \tag{4.4}$$

is true provided that $\{D_n\}$ is an approximating family for D and $G_n \subset D_n\backslash\{0\}$. Then, Theorem 4.10$(a)$ improves [6, Theorem 3.7(c)], and it follows that

$$\limsup_{\varepsilon\to 0} \mathrm{He}(f,S,C_n,\varepsilon) \subset \limsup_{\varepsilon\to 0} \mathrm{Be}(f,S,C_n,\varepsilon) \subset \mathrm{E}(f,S,D),$$

for every $n \in \mathbb{N}$ (we have applied Remark 2.10(c) in the first inclusion and [6, Theorem 3.7(c)] in the second one). But actually, in (4.4) we have shown that the upper limit of Henig approximate efficient solutions is included in the set of exact proper efficient solutions $\mathrm{He}(f,S,D)$, which is a more precise estimation than $\mathrm{E}(f,S,D)$.

Furthermore, if $\{D_n\} \in \bigcap_{x\in S} S(D,x)$, then Theorem 4.10$(a)$ characterizes the set of Henig proper efficient solutions of problem (VOP) in terms of limits of Henig approximate proper efficient solutions when the error tends to zero.

(c) By means of Theorem 4.10(b) we see that for $q \in D\backslash\{0\}$ and $\varepsilon > 0$ small enough, the notion given by El Maghri [3], and consequently by Rong [23] (see Remark 2.10(d) and Theorem 3.1) provide a set of approximate proper solutions that tend to weak efficient solutions. However, if our aim is to provide a suitable approximation of the proper efficient set, we have to consider a more restrictive approximation set than $q + D$, as for instance, the sets $C_n = G_n + D_n$, with $G_n \subset D_n\backslash\{0\}$ and n big enough, as it was proved in part (a) (take also into account Remark 4.4).

In the following example, we illustrate the results.

Example 4.12. Let $X = Y = \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity function on $\mathbb{R}^2$, $S = \mathbb{R}^2_+ \cap \mathcal{Q}^c$, where $\mathcal{Q}$ denotes the open square $(0,1)\times(0,1)$ and $D = P = \{(x_1,x_2)^t \in \mathbb{R}^2_+ : x_2 \geq x_1\}$. It is easy to see that

$$\mathrm{He}(f,S,P) = \mathrm{E}(f,S,P) = \{(x_1,1)^t \in \mathbb{R}^2 : 0 \leq x_1 < 1\} \cup \{(x_1,0)^t \in \mathbb{R}^2 : x_1 \geq 1\},$$

and $\mathrm{WE}(f,S,P) = \mathrm{bd}\, S$.

Let us consider $\varepsilon = 0.1$ and $q = (1,2)^t \in P$. In Figure 1 we have represented the set $\mathrm{He}(f,S,q+P,0.1)$ in dark grey.

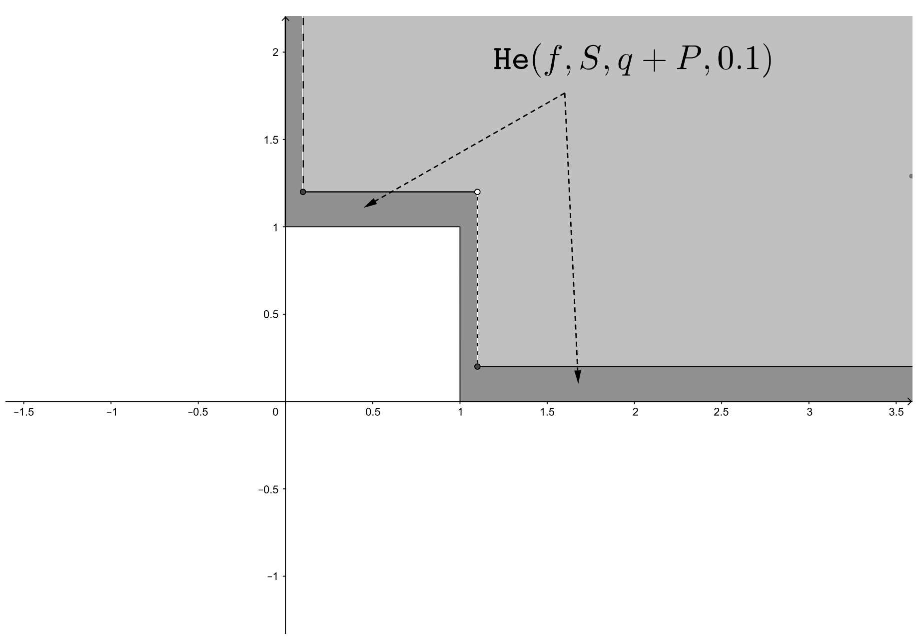

FIG. 1. $\mathrm{He}(f,S,q+P,0.1)$

As it can be observed, this set does not provide a suitable approximation of the proper efficient set (which, in this case, is also equal to the efficient set), since we can find approximate proper solutions as far as one wants from $\mathrm{He}(f,S,P)$.

Indeed, every point $(x_1,x_2)^t \in \mathbb{R}^2$, with $0 \leq x_1 < 0.1$ and $x_2 \geq 1$ is an approximate proper efficient solution, and the distance from such a point to the efficient set tends to infinity when x_2 goes to infinity.

In this case, what we obtain is a good approximation of the weak efficient set.

On the other hand, it is clear that the cone P is polyhedral, constructed through the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

We know from Theorem 2.4 that $\{P_n\} \in \bigcap_{x \in S} \mathcal{S}(P, x)$. If we consider, for instance, $n = 10$, it follows that

$$P_{10} = \left\{ (x_1, x_2)^t \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0.2 \\ 0 & 1.2 \\ -1 & 1.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2_+ \right\}.$$

So take now $C_{10} = q + P_{10}$. The set $\mathrm{He}(f, S, C_{10}, 0.1)$ is illustrated in Figure 2. As it can be observed, it provides a good approximation of the proper efficient set. Indeed, every approximate proper solution is close to the proper efficient set, which is precisely the property studied in Theorem 4.3(e) and Remark 4.4(a),(b).

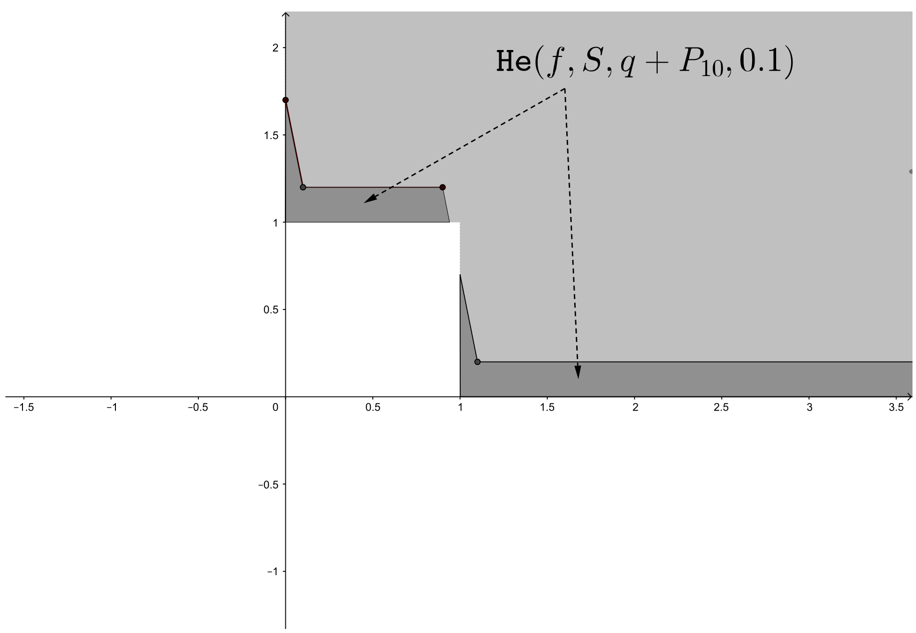

FIG. 2. $\mathrm{He}(f, S, q + P_{10}, 0.1)$

Although it is clear from Theorem 4.3(e), we underline that set $\mathrm{He}(f, S, C_{10}, 0.1)$ does not contain any point of the set

$$\{(0, x_2)^t \in \mathbb{R}^2 : x_2 > 1\} \cup \{(1, x_2)^t \in \mathbb{R}^2 : 0 < x_2 \leq 1\},$$

that represents the collection of weak efficient solutions that are not efficient solutions. This situation can be visualized better in Figure 3, in which we have improved the accuracy by considering $\varepsilon = 0.03$.

Of course, the higher the value of n and the smaller the value of ε, the better the approximation of $\mathrm{He}(f, S, q + P_n, \varepsilon)$ to the proper efficient set (see Theorem 4.10(a)).

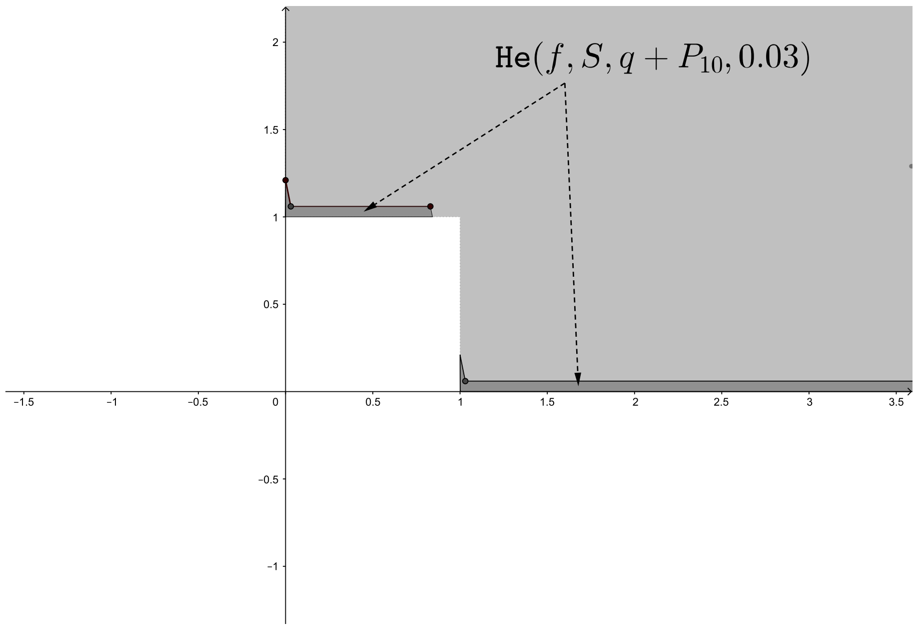

FIG. 3. He$(f, S, q + P_{10}, 0.03)$

5. Conclusions. We have studied the limit behaviour when the precision goes to zero of approximate proper efficient solutions of a vector optimization problem with an arbitrary closed pointed convex ordering cone. These solutions are defined by means of a set that approximates the ordering cone. For different choices of the approximating set, we have obtained sufficient conditions for approximate proper solutions to tend to exact weak/efficient/proper solutions when the precision error goes to zero.

Moreover, we have guaranteed the convergence of the approximate proper solutions to the exact proper efficient solutions for different families of approximating sets.

The main results of this work are useful to characterize approximate proper solutions of the vector optimization problem through scalarization. In this case, one could obtain suitable approximate proper efficient solutions by solving scalar optimization problems.

Thus, this research is the theoretical basis of a forthcoming paper, where we will address with scalarization processes, paying attention to some interesting settings from a computational point of view, as the nonconvex finite dimensional vector optimization problems with a polyhedral ordering cone.

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