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New notions of proper efficiency in set optimization with the set criterion

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Abstract In this paper, we introduce new notions of proper efficiency in the sense of Henig for a set optimization problem by using the set criterion of solution. The relationships between them are studied. Also, we compare these concepts with the homologous ones given by considering the vector criterion. Finally, a Lagrange multiplier rule for Henig proper solutions of a set optimization problem with a cone constraint is obtained under convexity hypotheses. Illustrative examples are also given.

Keywords Set optimization · Set criterion of solution · Vector criterion of solution · Henig proper efficiency · Lagrange multipliers

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1 Introduction

When we solve a vector optimization problem, the set of efficient points is sometimes very big and because of that it may contain anomalous or non desirable points. The notions of proper efficiency appear in the literature with the aim of providing a suitable selection of efficient points, that satisfy better properties, following some criterion.

The first notion of proper efficiency was given by Kuhn and Tucker [27] for multiobjective optimization problems and modified later by Geoffrion [7]. Several years later, Borwein [4] introduced a concept of proper efficiency for vector optimization problems, and in the same setting Benson [3] defined a notion of proper efficiency that extends the concept by Geoffrion and implies the notion by Borwein.

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On the other hand, Henig [15] introduced a concept of proper efficiency for vector optimization, more restrictive than the one given by Benson, and based on the idea of replacing the ordering cone by a bigger one, with non empty interior. Thus, the set of proper efficient points in the sense of Henig is easier to obtain and satisfies better properties, since it consists, basically, in a set of weak efficient points with respect to dilating cones.

One of the main properties of the proper efficient solutions in the sense of Henig is that, under convexity hypotheses, they can be characterized through linear scalarization (see [11,15]), which facilitates their calculus. Moreover, since this notion is based on cones with non empty interior, these solutions can be characterized through nonlinear scalarization as well, for which no convexity conditions are required (see, for example, [12]). Because of that, Henig proper efficiency has resulted to be a fruitful notion in vector optimization, as it is proved by the numerous papers dealing with it (see, for instance, [6,9,11–13]).

If we extend the framework to set-valued optimization problems, we find that there exist mainly two types of criteria to define a solution of these problems: the vector criterion and the set criterion. The first one (see [21,32]) consists in obtaining the efficient points of the image of the set-valued map, considering the image set as a set of elements of the final space, and taking into account the ordering given in the final space.

The second one, the set criterion ([28,29]) requires to establish a set order relation between the sets of the final space. Then, by means of this criterion one compares values of the set-valued map (which are sets) and chooses the minimal ones taking into account the set order relation. Then, this criterion seems to be more natural to handle with set-valued problems. A set-valued optimization problem treated with this criterion is called a set optimization problem.

Set-valued optimization problems have become interesting since several years ago, due to their multiple applications in different fields of research, as mathematical economics and finance, optimal control and viability theory (see, for instance, [2,14,21]).

In the literature, one can find some references about proper efficiency for a set-valued optimization problem with the vector criterion (see, for instance, [30,31]), but to the best of our knowledge, proper efficiency with the set criterion has not been considered. The aim of this work is to extend the concept by Henig to the set optimization framework. Specifically, we define two notions of proper efficiency in the sense of Henig for a set optimization problem, i.e., with the set criterion, and compare them with their counterparts defined with the vector criterion. Also, we provide a Lagrange multiplier rule for Henig proper solutions of a convex set optimization problem with cone constraints, by using the oriented distance of Hiriart-Urruty [18] as scalarizing functional.

The paper is structured as follows. In Section 2 we state the framework, the basic results and previous notions that we need along the paper. In Section 3 we introduce two notions of proper efficiency in the sense of Henig for a set optimization problem with the set criterion, we study some properties and compare these concepts with the analogous notions by using the vector criterion. Then, in Section 4 we establish a Lagrange multiplier rule for Henig proper solutions of a constrained set optimization problem, under convexity conditions. As an application, we derive a multiplier rule for Henig proper solutions when the set optimization problem reduces to a vector one. Several illustrative examples are also provided. Finally, in Section 5 we state the conclusions.

2 Preliminaries and basic results

Let Y be a real normed space. Given a subset $A \subset Y$, we denote the interior, the closure, the boundary and the cone generated by A as $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$ and $\operatorname{cone} A$, respectively. It is said that A is solid if $\operatorname{int} A \neq \emptyset$. Let $\mathcal{P}_0(Y)$ be the set of all nonempty subsets of Y.

Given $y \in Y$ and $A \subset Y$, we recall that the distance of y to A is given by $d(y, A) := \inf_{x\in A} \|x - y\|$, being $+\infty$ if $A = \emptyset$. We denote by $\mathbb{B}$ (resp., $\bar{\mathbb{B}}$) the open (resp., closed) unit ball in Y, and $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Throughout the paper, $K \subset Y$ is a proper ($\{0\} \neq K \neq Y$) pointed ($K \cap (-K) = \{0\}$) closed convex cone, not necessarily solid. In Y, a partial order $\leq_K$ (reflexive, transitive and antisymmetric) induced by K is defined as usual

$$x, y \in Y,\ x \leq_K y \Leftrightarrow y - x \in K.$$

Moreover, if K is solid, it is defined $x \ll_K y$ if $y - x \in \operatorname{int} K$.

It is said that a nonempty subset A of Y is K-proper if $A + K \neq Y$, K-closed if $A+K$ is closed, K-bounded if there exists $t > 0$ such that $A \subset t\mathbb{B}+K$, and K-compact if any cover of A of the form $\{U_\alpha + K : \alpha \in I,\ U_\alpha \text{ is open}\}$ admits a finite subcover.

Remark 2.1 Every K-compact set is K-closed and K-bounded (see [32, Chapter 1, Proposition 3.3]). Every K-bounded set is also K-proper.

Let Y^* be the topological dual space of Y, and denote by $\langle y^*, y\rangle$ the duality pairing between Y^* and Y. Recall that the positive polar cone and the strictly positive set to K are defined, respectively, by

$$K^+ := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0,\ \forall y \in K\},$$
$$K^{s+} := \{y^* \in Y^* : \langle y^*, y\rangle > 0,\ \forall y \in K \setminus \{0\}\}.$$

We say that a convex set $\Theta \subset Y$ is a base of K if $K = \operatorname{cone}\Theta$ and $0 \notin \operatorname{cl}\Theta$. It is known that K has a base if and only if $K^{s+} \neq \emptyset$ (see, for instance, Jahn [21]).

In the sequel, we always assume that K has a base Θ. Set

$$\delta := \inf\{\|y\| : y \in \Theta\} > 0.$$

For each scalar $\eta \in (0, \delta)$ we associate to K a cone

$$K_\eta := \operatorname{cl}\operatorname{cone}(\Theta + \eta\bar{\mathbb{B}}).$$

This cone, known as the Henig dilating cone [15], plays an important role in the study of proper efficiency.

Remark 2.2 (i) By [5, Theorem 1.1(1)], we have that $K_\eta = \operatorname{cone}\operatorname{cl}(\Theta + \eta\bar{\mathbb{B}})$. Also, it follows that

$$\operatorname{cl}(\Theta + \eta\bar{\mathbb{B}}) = B_\eta := \{y \in Y : d(y, \Theta) \leq \eta\}, \tag{1}$$

so $K_\eta = \operatorname{cone} B_\eta$. The proof of equality (1) can be found in [23, Lemma 2.9], but we add it below for the convenience of the reader.

Indeed, let $y \in \Theta + \eta\bar{\mathbb{B}}$. Then, there exists $\theta \in \Theta$ such that $y \in \theta + \eta\bar{\mathbb{B}}$, so $d(y, \theta) \leq \eta$ and then $d(y, \Theta) \leq \eta$. Hence, $\Theta + \eta\bar{\mathbb{B}} \subset B_\eta$, so we deduce that $\operatorname{cl}(\Theta + \eta\bar{\mathbb{B}}) \subset B_\eta$, since B_η is closed.

Reciprocally, let $y \in B_\eta$ and suppose by reasoning to the contrary that $y \notin \operatorname{cl}(\Theta + \eta\bar{\mathbb{B}})$. Then, there exists $\alpha > 0$ such that

$$d(y, \Theta + \eta\bar{\mathbb{B}}) \geq \alpha. \tag{2}$$

Fix $\theta \in \Theta$. We affirm that

$$d(y, \theta) \geq \alpha + \eta. \tag{3}$$

Otherwise, if $d(y, \theta) < \alpha + \eta$, there exists $b \in \mathbb{B}$ such that $y = \theta + (\alpha + \eta)b$. Therefore,

$$d(y, \theta + \eta b) = d(\theta + \alpha b + \eta b, \theta + \eta b) = \alpha\|b\| < \alpha,$$

and we would obtain a contradiction with (2).

Since (3) holds for all $\theta \in \Theta$, we deduce that $d(y, \Theta) = \inf_{\theta\in\Theta} d(y, \theta) \geq \alpha + \eta$, which is a contradiction, since $y \in B_\eta$.

As a consequence, the Henig dilating cones defined here coincide with the Henig dilating cones defined in [10, Lemma 3.2.51].

(ii) Let us also note that in the finite dimensional case, base Θ can be considered to be compact. Hence, in this case, $\Theta + \eta\bar{\mathbb{B}}$ is closed, and then $K_\eta = \operatorname{cone}(\Theta + \eta\bar{\mathbb{B}})$.

The properties of the Henig dilating cone are collected in the following result from Göpfert et al. [10, Lemma 3.2.51] (see also Gong [8, Lemma 2.1]).

Lemma 2.1 *(i) The Henig dilating cone K_η is a solid pointed closed convex cone for every $\eta \in (0, \delta)$.*

(ii) If $0 < \eta_1 < \eta_2 < \delta$, then $K \setminus \{0\} \subset K_{\eta_1} \setminus \{0\} \subset \operatorname{int} K_{\eta_2}$.

We denote

$$K^\Delta(\Theta) := \{y^* \in K^{s+} : \exists t > 0 \text{ such that } \langle y^*, y\rangle \geq t \ \forall y \in \Theta\}.$$

By a separation theorem of convex sets, we know $K^\Delta(\Theta) \neq \emptyset$. The set $K^\Delta(\Theta)$ was introduced by Zheng in [36]. It is clear that $K^\Delta(\Theta) \cup \{0\}$ is a pointed convex cone, and $K^+ + K^\Delta(\Theta) \subset K^\Delta(\Theta) \subset K^{s+}$. It is known that $K^\Delta(\Theta)$ is in general strictly contained in K^{s+} (see [9]). In the following lemma we collect some properties about the positive polar cone to K_η (see [33, Lemma 5.1]).

Lemma 2.2 *The following properties hold.*
(i) $(\bigcup_{0<\eta<\delta} K_\eta^+) \setminus \{0\} = K^\Delta(\Theta)$*.*
(ii) $\operatorname{int}(K^+) \subset K^\Delta(\Theta)$*; and when* Θ *is bounded, then* $\operatorname{int}(K^+) = K^\Delta(\Theta)$*.*

Hiriart-Urruty [18] introduced the next nonlinear scalarization function.
Let $A \subset Y$. The oriented distance $D(\cdot, A) : Y \to \bar{\mathbb{R}}$ is defined as follows:

$$D(y, A) := d(y, A) - d(y, Y \backslash A).$$

If $A, B \in \mathcal{P}_0(Y)$, we define the following function from $\mathcal{P}_0(Y) \times \mathcal{P}_0(Y)$ into $\bar{\mathbb{R}}$:

$$\mathbb{D}_K^{si}(A, B) := \sup_{b \in B} \inf_{a \in A} D(a - b, -K).$$

This scalarization function has been studied, for example, in [24, 25].

2.1 Efficiency and proper efficiency notions for sets

Let A be a nonempty proper subset of Y. In this paper we consider the following concepts of efficiency [12, 13, 15, 34, 35].

We denote

$$\mathcal{H}_K := \{K' \subset Y : K' \text{ is a pointed solid convex cone and } K \setminus \{0\} \subset \operatorname{int} K'\}.$$

Definition 2.1 Let $a_0 \in A$. It is said that a_0 is
(i) an efficient point of A with respect to (wrt) K ($a_0 \in \operatorname{Min}(A, K)$) if $(A - a_0) \cap (-K \setminus \{0\}) = \emptyset$,
(ii) a weak efficient point of A wrt K ($a_0 \in \operatorname{wMin}(A, K)$) if $(A - a_0) \cap (-\operatorname{int} K) = \emptyset$,
(iii) a Henig proper efficient point of A wrt K ($a_0 \in \operatorname{He}(A, K)$) if there exists $K' \in \mathcal{H}_K$ such that $a_0 \in \operatorname{Min}(A, K')$,
(iv) a Henig proper efficient point of A wrt Θ ($a_0 \in \operatorname{He}(A, \Theta)$) if there exists $\eta \in (0, \delta)$ such that $a_0 \in \operatorname{Min}(A, K_\eta)$.

From now on, to simplify, we will just write Henig proper efficient points instead of Henig proper efficient points wrt to a base Θ. Also, we will assume that $\operatorname{int} K \neq \emptyset$ whenever we deal with weak efficient points.

From the definition, one has

$$\operatorname{He}(A, \Theta) \subset \operatorname{He}(A, K) \subset \operatorname{Min}(A, K) \subset \operatorname{wMin}(A, K) \tag{4}$$

(the first inclusion follows from the fact that $K_\eta \in \mathcal{H}_K$ for all $\eta \in (0, \delta)$ by Lemma 2.1).

Remark 2.3 (i) One has

$$\operatorname{He}(A, K) = \bigcup_{K' \in \mathcal{H}_K} \operatorname{Min}(A, K') = \bigcup_{K' \in \mathcal{H}_K} \operatorname{wMin}(A, K').$$

Indeed, the first equality is by the definition. As $\mathrm{Min}(A,K') \subset \mathrm{wMin}(A,K')$, we only have to prove that $\bigcup_{K'\in\mathcal{H}_K} \mathrm{wMin}(A,K') \subset \bigcup_{K'\in\mathcal{H}_K} \mathrm{Min}(A,K')$. But this is clear since $\mathrm{wMin}(A,K') = \mathrm{Min}(A,K_1)$, where $K_1 := (\mathrm{int}\, K') \cup \{0\} \in \mathcal{H}_K$ whenever $K' \in \mathcal{H}_K$.

(ii) To define $\mathcal{H}_K$ we have considered that K' is pointed (as in [26, Definition 2.4.4(f)]), but in some papers (see, f.e., [15,35]) the pointedness of K' is not required. Let us see that $\mathrm{He}(A,K)$ does not change if we remove the pointedness condition on K' and we ask for the properness of K'. Indeed, let

$$\tilde{\mathcal{H}}_K := \left\{\tilde{K} \subset Y : \tilde{K} \text{ is a proper solid convex cone and } K\setminus\{0\} \subset \mathrm{int}\,\tilde{K}\right\},$$

and let us prove that

$$\bigcup_{K'\in\mathcal{H}_K} \mathrm{Min}(A,K') = \bigcup_{\tilde{K}\in\tilde{\mathcal{H}}_K} \mathrm{Min}(A,\tilde{K}).$$

The inclusion "$\subset$" is obvious as $\mathcal{H}_K \subset \tilde{\mathcal{H}}_K$. To prove the converse inclusion, let $a_0 \in \bigcup_{\tilde{K}\in\tilde{\mathcal{H}}_K} \mathrm{Min}(A,\tilde{K})$. Then there exists $\tilde{K} \in \tilde{\mathcal{H}}_K$ such that $a_0 \in \mathrm{Min}(A,\tilde{K}) \subset \mathrm{wMin}(A,\tilde{K})$. Let $K_1 := \mathrm{int}\,\tilde{K} \cup \{0\}$. It follows that $a_0 \in \mathrm{Min}(A,K_1)$ and, moreover, K_1 is pointed and $K\setminus\{0\} \subset \mathrm{int}\,\tilde{K} \subset \mathrm{int}\,K_1$. Therefore, $K_1 \in \mathcal{H}_K$ and consequently $a_0 \in \bigcup_{K'\in\mathcal{H}_K} \mathrm{Min}(A,K')$.

The following result plays a crucial role to obtain properties of Henig proper efficient points. It allows us to describe, equivalently, a Henig proper efficient point as a weak efficient point wrt some Henig dilating cone.

Proposition 2.1 *One has*

$$\mathrm{He}(A,\Theta) = \bigcup_{\eta\in(0,\delta)} \mathrm{Min}(A,K_\eta) = \bigcup_{\eta\in(0,\delta)} \mathrm{wMin}(A,K_\eta).$$

Proof The first equality is by definition. As $\mathrm{Min}(A,K_\eta) \subset \mathrm{wMin}(A,K_\eta)$, we only have to prove $\bigcup_{\eta\in(0,\delta)} \mathrm{wMin}(A,K_\eta) \subset \bigcup_{\eta\in(0,\delta)} \mathrm{Min}(A,K_\eta)$. Let $a_0 \in \mathrm{wMin}(A,K_\eta)$ for some $\eta \in (0,\delta)$, then $(A-a_0)\cap(-\mathrm{int}\,K_\eta) = \emptyset$. By applying Lemma 2.1(ii), and choosing $\bar{\eta} \in (0,\eta)$, one has $K_{\bar{\eta}}\setminus\{0\} \subset \mathrm{int}\,K_\eta$, and so $(A-a_0)\cap(-K_{\bar{\eta}}\setminus\{0\}) = \emptyset$, which means that $a_0 \in \mathrm{Min}(A,K_{\bar{\eta}})$.

2.2 Efficiency and proper efficiency notions for set-valued maps

Let X be a linear space. Given a set-valued map $F : X \rightrightarrows Y$, its graph is $\mathrm{gr}\,F := \{(x,y) \in X\times Y : y \in F(x)\}$, its epigraph is $\mathrm{epi}\,F := \{(x,y) \in X\times Y : y \in F(x)+K\}$, and its profile (or epigraphical) map is the set-valued map $F_K : X \rightrightarrows Y$ defined by $F_K(x) := F(x)+K$, which is also denoted by $F+K$. It is clear that $\mathrm{gr}\,F_K = \mathrm{epi}\,F$.

Whenever 'N' denotes some property of sets in Y, it is said that F is 'N'-valued if $F(x)$ has the property 'N' for every $x \in X$.

Let us recall some concepts of minimizers of a set-valued map $F : X \rightrightarrows Y$ with the vector criterion. Note that the concepts of efficient points of a set in Definition 2.1 naturally induce the following concepts of minimizers of F. Given $\emptyset \neq S \subset X$, we set $F(S) := \cup_{x \in S} F(x)$.

Definition 2.2 Let $x \in S$. We say that x is a

(i) minimizer of F wrt K, denoted $x \in \mathrm{Min}(F, S, K)$, if there exists $y \in F(x)$ such that $y \in \mathrm{Min}(F(S), K)$,

(ii) weak minimizer of F wrt K, denoted $x \in \mathrm{wMin}(F, S, K)$, if there exists $y \in F(x)$ such that $y \in \mathrm{wMin}(F(S), K)$,

(iii) Henig proper minimizer of F wrt K, denoted $x \in \mathrm{He}(F, S, K)$, if there exists $y \in F(x)$ such that $y \in \mathrm{He}(F(S), K)$,

(iv) Henig proper minimizer of F wrt Θ, denoted $x \in \mathrm{He}(F, S, \Theta)$, if there exists $y \in F(x)$ such that $y \in \mathrm{He}(F(S), \Theta)$.

It is easy to see that

$$\mathrm{He}(F, S, \Theta) \subset \mathrm{He}(F, S, K) \subset \mathrm{Min}(F, S, K) \subset \mathrm{wMin}(F, S, K).$$

Remark 2.4 One has

$$\mathrm{He}(F, S, K) = \bigcup\nolimits_{K' \in \mathcal{H}_K} \mathrm{Min}(F, S, K') = \bigcup\nolimits_{K' \in \mathcal{H}_K} \mathrm{wMin}(F, S, K'), \quad (5)$$

$$\mathrm{He}(F, S, \Theta) = \bigcup\nolimits_{\eta \in (0,\delta)} \mathrm{Min}(F, S, K_\eta) = \bigcup\nolimits_{\eta \in (0,\delta)} \mathrm{wMin}(F, S, K_\eta). \quad (6)$$

Indeed, (5) is a consequence of the definition and Remark 2.3(i), and (6) follows from Proposition 2.1.

Next, we recall the so-called lower (and strict lower) set less order relation. Given $A, B \in \mathcal{P}_0(Y)$, we consider the following set relations:

- $A \leq^l_K B \Leftrightarrow \forall b \in B, \exists a \in A : a - b \in -K \Leftrightarrow B \subset A + K$.
- $A \ll^l_K B \Leftrightarrow \forall b \in B, \exists a \in A : a - b \in -\operatorname{int} K \Leftrightarrow B \subset A + \operatorname{int} K$.

Remark 2.5 When A and B are singleton, $A = \{a\}$ and $B = \{b\}$, we have $\{a\} \leq^l_K \{b\}$ if and only if $a \leq_K b$, and $\{a\} \ll^l_K \{b\}$ if and only if $a \ll_K b$.

Let S be a non-empty subset of X. We consider the following set optimization problem

$$\leq^l_K - \min F(x) \quad \text{subject to } x \in S. \qquad \text{(SOP)}$$

In the following, we recall some concepts of minimal solution using the set criterion and the set relations stated above (see, for instance, [17, 22, 24, 28]).

Definition 2.3 Let $x_0 \in S$. It is said that x_0 is a

(i) $\leq^l_K$-minimal solution of (SOP), denoted $x_0 \in \mathrm{Min}_l(F, S, K)$, if $F(x) \leq^l_K F(x_0)$ for some $x \in S$ implies $F(x_0) \leq^l_K F(x)$,

(ii) weak $\leq^l_K$-minimal solution of (SOP), denoted $x_0 \in \mathrm{wMin}_l(F, S, K)$, if $F(x) \ll^l_K F(x_0)$ for some $x \in S$ implies $F(x_0) \ll^l_K F(x)$.

Of course, if we deal with weak $\leq^l_K$-minimal solutions, we suppose that K is solid.

By [17, Proposition 2.7(i)] we know that

$$\mathrm{Min}_l(F, S, K) \subset \mathrm{wMin}_l(F, S, K). \quad (7)$$

3 Henig proper solutions of (SOP)

In this section we introduce the notion of Henig proper solution of (SOP) with the set criterion. We also study its basic properties and relationships with the Henig proper solution defined when the vector criterion is considered. As far as we know, the notion of Henig proper efficiency with the set criterion has not been considered in the literature up to now.

We denote $\mathrm{E}_l(x_0) := \{x \in S : F(x) \leq^l_K F(x_0) \text{ and } F(x_0) \leq^l_K F(x)\}$. Let us recall that $x \in \mathrm{E}_l(x_0)$ if and only if $F(x_0) + K = F(x) + K$.

Definition 3.1 We say that $x_0 \in S$ is a

(i) $\leq^l_K$-Henig proper solution of (SOP), denoted $x_0 \in \mathrm{He}_l(F, S, K)$, if there exists $K' \in \mathcal{H}_K$ such that $x_0 \in \mathrm{Min}_l(F, S, K')$, i.e.,

$$x \in S,\ F(x) \leq^l_{K'} F(x_0) \ \Rightarrow\ F(x_0) \leq^l_{K'} F(x),$$

(ii) strict $\leq^l_K$-Henig proper solution of (SOP), denoted $x_0 \in \mathrm{sHe}_l(F, S, K)$, if there exists $K' \in \mathcal{H}_K$ such that there is no $x \in S \setminus \mathrm{E}_l(x_0)$ with $F(x) \leq^l_{K'} F(x_0)$, i.e.,

$$F(x) \not\leq^l_{K'} F(x_0), \quad \forall x \in S \setminus \mathrm{E}_l(x_0), \tag{8}$$

(iii) weak $\leq^l_K$-Henig proper solution of (SOP) wrt Θ, denoted $x_0 \in \mathrm{wHe}_l(F, S, \Theta)$, if there exists $\eta \in (0, \delta)$ such that $x_0 \in \mathrm{wMin}_l(F, S, K_\eta)$, i.e.,

$$x \in S,\ F(x) \ll^l_{K_\eta} F(x_0) \ \Rightarrow\ F(x_0) \ll^l_{K_\eta} F(x),$$

(iv) strict weak $\leq^l_K$-Henig proper solution of (SOP) wrt Θ, denoted $x_0 \in \mathrm{swHe}_l(F, S, \Theta)$, if there exists $\eta \in (0, \delta)$ such that there is no $x \in S \setminus \mathrm{E}_l(x_0)$ with $F(x) \ll^l_{K_\eta} F(x_0)$, i.e.,

$$F(x) \not\ll^l_{K_\eta} F(x_0), \quad \forall x \in S \setminus \mathrm{E}_l(x_0). \tag{9}$$

From the definition it is evident that

$$\begin{array}{l} \mathrm{He}_l(F, S, K) = \bigcup_{K' \in \mathcal{H}_K} \mathrm{Min}_l(F, S, K') \quad \text{and} \\ \mathrm{wHe}_l(F, S, \Theta) = \bigcup_{\eta \in (0,\delta)} \mathrm{wMin}_l(F, S, K_\eta). \end{array} \tag{10}$$

Remark 3.1 We can define in a similar way the following sets of Henig proper solutions:

$$\begin{aligned}
&\mathrm{wHe}_l(F, S, K) := \textstyle\bigcup_{K' \in \mathcal{H}_K} \mathrm{wMin}_l(F, S, K'),\\
&\mathrm{swHe}_l(F, S, K) := \{x_0 \in S : \exists K' \in \mathcal{H}_K \text{ such that } F(x) \not\ll^l_{K'} F(x_0)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall x \in S \setminus \mathrm{E}_l(x_0)\},\\
&\mathrm{He}_l(F, S, \Theta) := \textstyle\bigcup_{\eta \in (0,\delta)} \mathrm{Min}_l(F, S, K_\eta),\\
&\mathrm{sHe}_l(F, S, \Theta) := \big\{x_0 \in S : \exists \eta \in (0, \delta) \text{ such that } F(x) \not\leq^l_{K_\eta} F(x_0)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall x \in S \setminus \mathrm{E}_l(x_0)\big\}.
\end{aligned}$$

Let us note that none of them requires K to be solid. Taking into account that the cones $K_\eta \in \mathcal{H}_K$, the following inclusions are obvious:

$$\begin{array}{ll} \mathrm{He}_l(F,S,\Theta) \subset \mathrm{He}_l(F,S,K), & \mathrm{sHe}_l(F,S,\Theta) \subset \mathrm{sHe}_l(F,S,K), \\ \mathrm{wHe}_l(F,S,\Theta) \subset \mathrm{wHe}_l(F,S,K), & \mathrm{swHe}_l(F,S,\Theta) \subset \mathrm{swHe}_l(F,S,K). \end{array} \tag{11}$$

The following inclusions are also obvious (Remark 3.1 and conditions (10) and (7) are used to prove the first and third inclusions):

$$\begin{array}{ll} \mathrm{He}_l(F,S,K) \subset \mathrm{wHe}_l(F,S,K), & \mathrm{sHe}_l(F,S,K) \subset \mathrm{swHe}_l(F,S,K), \\ \mathrm{He}_l(F,S,\Theta) \subset \mathrm{wHe}_l(F,S,\Theta), & \mathrm{sHe}_l(F,S,\Theta) \subset \mathrm{swHe}_l(F,S,\Theta). \end{array} \tag{12}$$

We say that each pair of sets in the four inclusions in (11) are analogous, in the following sense: most of the results that we will establish in this section for a set in Definition 3.1 will be also true for the analogous set with a very similar proof (see Remark 3.2). Specially for this reason, we will mainly consider the sets introduced in Definition 3.1.

Next, we are going to study some relationships between the different sets of $\leq^l_K$-Henig proper solutions. First of all, we state some basic inclusions (Proposition 3.1). Second, to establish other interesting inclusions (Proposition 3.2), we need some previous results (Lemmas 3.1-3.3).

Proposition 3.1 *The following inclusions are fulfilled:*

$$\mathrm{sHe}_l(F,S,K) \subset \mathrm{He}_l(F,S,K) \quad \textit{and} \quad \mathrm{sHe}_l(F,S,K) \subset \mathrm{Min}_l(F,S,K), \tag{13}$$

$$\mathrm{swHe}_l(F,S,\Theta) \subset \mathrm{wHe}_l(F,S,\Theta), \tag{14}$$

and if K is solid then

$$\mathrm{swHe}_l(F,S,\Theta) \subset \mathrm{wMin}_l(F,S,K). \tag{15}$$

Proof Let $x_0 \in \mathrm{sHe}_l(F,S,K)$, then (8) is satisfied for some $K' \in \mathcal{H}_K$. Let us prove that

$$x_0 \in \mathrm{Min}_l(F,S,K_1) \tag{16}$$

for all convex cone K_1 with $K \subset K_1 \subset K'$. Indeed, let $x \in S$ satisfying $F(x) \leq^l_{K_1} F(x_0)$. Then, by definition

$$F(x_0) \subset F(x) + K_1 \subset F(x) + K', \tag{17}$$

i.e., $F(x) \leq^l_{K'} F(x_0)$. As $x \in S$, in view of (8) it follows that $x \in \mathrm{E}_l(x_0)$. From here,

$$F(x_0) + K = F(x) + K. \tag{18}$$

Adding K to the first inclusion in (17) we have $F(x_0) + K \subset F(x) + K + K_1$. Using (18) we derive that $F(x) + K \subset F(x_0) + K + K_1 \subset F(x_0) + K_1$ since $K \subset K_1$ and $K_1 + K_1 \subset K_1$. As $F(x) \subset F(x) + K$, we deduce that $F(x) \subset F(x_0) + K_1$, which means that $F(x_0) \leq^l_{K_1} F(x)$, and therefore (16) is proved.

Now, the inclusions in (13) follow using (16) with $K_1 = K'$ and $K_1 = K$, respectively.

The inclusions (14) and (15) are proved using the same ideas. Indeed, let $x_0 \in \mathrm{swHe}_l(F, S, \Theta)$, then (9) holds for some K_η with $\eta \in (0, \delta)$. Let us prove that

$$x_0 \in \mathrm{wMin}_l(F, S, K_1) \tag{19}$$

for all solid convex cone K_1 with $K \subset K_1 \subset K_\eta$. Indeed, let $x \in S$ satisfying $F(x) \ll^l_{K_1} F(x_0)$. Then, by definition

$$F(x_0) \subset F(x) + \operatorname{int} K_1 \subset F(x) + \operatorname{int} K_\eta, \tag{20}$$

i.e., $F(x) \ll^l_{K_\eta} F(x_0)$. In view of (9) it follows that $x \in \mathrm{E}_l(x_0)$, and so, (18) holds. Adding K to (20) we have $F(x_0) + K \subset F(x) + K + \operatorname{int} K_1$. Using (18) we derive that $F(x) + K \subset F(x_0) + K + \operatorname{int} K_1 \subset F(x_0) + \operatorname{int} K_1$. As $F(x) \subset F(x) + K$, we get $F(x) \subset F(x_0) + \operatorname{int} K_1$, i.e., $F(x_0) \ll^l_{K_1} F(x)$, and therefore (19) is proved. Now, the inclusions (14) and (15) follow using (19) with $K_1 = K_\eta$ and $K_1 = K$, respectively, this latter when K is solid.

The next example shows that in general $\mathrm{wHe}_l(F, S, \Theta) \not\subset \mathrm{wMin}_l(F, S, K)$, $\mathrm{swHe}_l(F, S, \Theta) \neq \mathrm{wHe}_l(F, S, \Theta)$ and $\mathrm{swHe}_l(F, S, \Theta) \not\subset \mathrm{Min}_l(F, S, K)$.

Example 3.1 Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$, $S := \{0, 1\}$, $K := \mathbb{R}^2_+ := \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0,\ y_2 \geq 0\}$, $\Theta := \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 + y_2 = 2\}$ and so $\delta = \sqrt{2}$.

(a) Let $F(0) := \{(y, y) : y > 0\}$ and $F(1) := \{(0, y) : y > 0\}$. Then, for each $\eta \in (0, \delta)$, we have that

$$F(0) + \operatorname{int} K_\eta = F(1) + \operatorname{int} K_\eta = \operatorname{int} K_\eta,$$

so $F(0) \ll^l_{K_\eta} F(0)$, $F(1) \ll^l_{K_\eta} F(0)$ and $F(0) \ll^l_{K_\eta} F(1)$. Hence, it follows that $0 \in \mathrm{wHe}_l(F, S, \Theta)$. However, $0 \notin \mathrm{wMin}_l(F, S, K)$, since $F(1) \ll^l_K F(0)$, but $F(0) \not\ll^l_K F(1)$. Note also that $0 \notin \mathrm{swHe}_l(F, S, \Theta)$ (observe that $0 \in \mathrm{E}_l(0)$, but $1 \notin \mathrm{E}_l(0)$, since $F(1) \not\subseteq F(0) + K = \operatorname{int} \mathbb{R}^2_+$, and then $F(0) \not\leq_K F(1)$), and so

$$\mathrm{swHe}_l(F, S, \Theta) \neq \mathrm{wHe}_l(F, S, \Theta).$$

(b) Now, let $F(0) := \{(y, y) : y \geq 0\}$ and $F(1) := \{(y_1, y_2) : y_1 + y_2 = 0\}$. Then in this case we have that $0 \in \mathrm{swHe}_l(F, S, \Theta) \setminus \mathrm{Min}_l(F, S, K)$.

This example also shows that in general $\mathrm{He}_l(F, S, \Theta)$ is different from $\mathrm{wHe}_l(F, S, \Theta)$ since $0 \in \mathrm{wHe}_l(F, S, \Theta) \setminus \mathrm{He}_l(F, S, \Theta)$, in contrast to what happens with the vector criterion (see (6)).

The same example also shows that $0 \in \mathrm{swHe}_l(F, S, \Theta) \setminus \mathrm{sHe}_l(F, S, \Theta)$. And although the cones in $\mathcal{H}_K$ are more flexible than the cones K_η, it is also true that $0 \in \mathrm{swHe}_l(F, S, K) \setminus \mathrm{sHe}_l(F, S, K)$.

Lemma 3.1 *Let $A \in \mathcal{P}_0(Y)$ and $K_1 \subset Y$ be a convex cone such that $K \subset K_1$. If A is K-compact, then A is K_1-compact.*

Proof Let $\{U_\alpha + K_1 : \alpha \in I,\, U_\alpha \text{ is open}\}$ a cover of A. Put $V_\alpha := U_\alpha + K_1$. One has

$$V_\alpha + K = U_\alpha + K_1 + K = U_\alpha + K_1 = V_\alpha.$$

Hence, $\{V_\alpha + K : \alpha \in I\}$ is a cover of A and $V_\alpha = \cup_{y\in K_1}(U_\alpha + y)$ is open. As A is K-compact we can extract a finite subcover, i.e., there exist $\alpha_1, \dots, \alpha_n \in I$ such that $A \subset \bigcup_{i=1}^n (V_{\alpha_i} + K) = \bigcup_{i=1}^n V_{\alpha_i} = \bigcup_{i=1}^n (U_{\alpha_i} + K_1)$. So, we have proved that A has a finite subcover of $\{U_\alpha + K_1 : \alpha \in I\}$, and therefore, A is K_1-compact.

Observe that if A is compact then A is K'-compact for all convex cone K', since in particular every cover of A of the form $\{U_\alpha + K' : \alpha \in I, U_\alpha \text{ open}\}$ admits a finite subcover.

Lemma 3.2 *(i) ([17, Lemma 2.6]) Assume K is solid and* $\mathrm{wMin}(F(x_0), K) \neq \emptyset$*. Then $x_0 \in \mathrm{wMin}_l(F, S, K)$ if and only if $x_0 \in S$ and there is not $x \in S$ such that $F(x) \ll_K^l F(x_0)$.*

(ii) Suppose that $\mathrm{Min}(F(x_0), K) \neq \emptyset$ *and that $F(x_0) \cap F(x) = \emptyset$, for all $x \in S\backslash\{x_0\}$. Then $x_0 \in \mathrm{Min}_l(F, S, K)$ if and only if $x_0 \in S$ and there is not $x \in S\backslash\{x_0\}$ such that $F(x) \leq_K^l F(x_0)$.*

Proof (ii) Let $x_0 \in \mathrm{Min}_l(F, S, K)$ and suppose by reasoning to the contrary that there is $x \in S\backslash\{x_0\}$ such that $F(x_0) \subset F(x) + K$. Then, we have $F(x_0) \subset F(x) + K\backslash\{0\}$. Otherwise, there would exist $y_0 \in F(x_0) \cap F(x)$, which is a contradiction.

Since $x_0 \in \mathrm{Min}_l(F, S, K)$ we have that $F(x) \subset F(x_0) + K$, so

$$F(x_0) \subset F(x) + K\backslash\{0\} \subset F(x_0) + K + K\backslash\{0\} = F(x_0) + K\backslash\{0\},$$

and then for any $y_0 \in F(x_0)$ we can find $z_0 \in F(x_0)$ such that $z_0 - y_0 \in -K\backslash\{0\}$, that contradicts $\mathrm{Min}(F(x_0), K) \neq \emptyset$.

Reciprocally, suppose that $x_0 \in S$ and there is no $x \in S\backslash\{x_0\}$ such that $F(x) \leq_K^l F(x_0)$. Then, by definition, $x_0 \in \mathrm{Min}_l(F, S, K)$, and the proof is complete.

Lemma 3.3 *(i) Assume $F(x_0)$ is K-compact. Then $x_0 \in \mathrm{wHe}_l(F, S, \Theta)$ if and only if $x_0 \in \mathrm{swHe}_l(F, S, \Theta)$.*

(ii) Assume $F(x_0)$ is K-compact and $F(x) \cap F(x_0) = \emptyset$ for all $x \in S\backslash\{x_0\}$. Then $x_0 \in \mathrm{He}_l(F, S, K)$ if and only if $x_0 \in \mathrm{sHe}_l(F, S, K)$.

Proof (i) Taking into account inclusion (14), we only have to prove the “only if” part. Let $x_0 \in \mathrm{wHe}_l(F, S, \Theta)$. Then there exists $\eta \in (0, \delta)$ such that $x_0 \in \mathrm{wMin}_l(F, S, K_\eta)$. As $F(x_0)$ is K-compact, by Lemma 3.1, we have that $F(x_0)$ is K_η-compact, and then $\mathrm{wMin}(F(x_0), K_\eta) \neq \emptyset$ by Luc [32, Chapter 2, Theorem 3.3 and Lemma 3.5]. By Lemma 3.2(i) it results that there exists no $x \in S$ such that $F(x) \ll_{K_\eta}^l F(x_0)$, and so, $x_0 \in \mathrm{swHe}_l(F, S, \Theta)$.

(ii) In view of the first inclusion in (13), we only have to prove the “only if” part. Let $x_0 \in \mathrm{He}_l(F, S, K)$, then $x_0 \in \mathrm{Min}_l(F, S, K')$ for some $K' \in \mathcal{H}_K$. As

$F(x_0)$ is K-compact, by Lemma 3.1, we have that $F(x_0)$ is K'-compact, and then by [32, Chapter 2, Corollary 3.7], we deduce that $\operatorname{Min}(F(x_0), K') \neq \emptyset$. Thus, as $F(x) \cap F(x_0) = \emptyset$ for all $x \in S \setminus \{x_0\}$ by assumption, in view of Lemma 3.2(ii) there is no $x \in S\setminus\{x_0\}$ such that $F(x) \leq^l_{K'} F(x_0)$, and so $x_0 \in \mathrm{sHe}_l(F, S, K)$.

The next result is an immediate consequence of Definition 3.1 and Lemma 3.3.

Proposition 3.2 *(i) Assume that F is K-compact valued. Then*

$$\mathrm{wHe}_l(F, S, \Theta) = \bigcup_{0<\eta<\delta} \mathrm{wMin}_l(F, S, K_\eta) = \mathrm{swHe}_l(F, S, \Theta).$$

(ii) If, in addition to (i), the following condition holds:

$$F(x_1) \cap F(x_2) = \emptyset \quad \forall x_1, x_2 \in S \text{ with } x_1 \neq x_2, \tag{21}$$

then

$$\mathrm{He}_l(F, S, K) = \bigcup_{K' \in \mathcal{H}_K} \mathrm{Min}_l(F, S, K') = \mathrm{sHe}_l(F, S, K).$$

Note that condition (21) is an extension of the notion of injective mapping to set-valued maps.

Remark 3.2 Following the same ideas developed in Propositions 3.1 and 3.2, the next inclusions can be proved if K is solid:

$$\begin{array}{ll} \mathrm{sHe}_l(F, S, \Theta) \subset \mathrm{He}_l(F, S, \Theta), & \mathrm{sHe}_l(F, S, \Theta) \subset \mathrm{Min}_l(F, S, K), \\ \mathrm{swHe}_l(F, S, K) \subset \mathrm{wHe}_l(F, S, K), & \mathrm{swHe}_l(F, S, K) \subset \mathrm{wMin}_l(F, S, K), \end{array}$$

and, if F is K-compact valued then

$$\mathrm{wHe}_l(F, S, K) = \mathrm{swHe}_l(F, S, K).$$

We summarize the inclusion relationships stated in Equations (7) and (12), Propositions 3.1 and 3.2 and Remark 3.2 in Fig. 1.

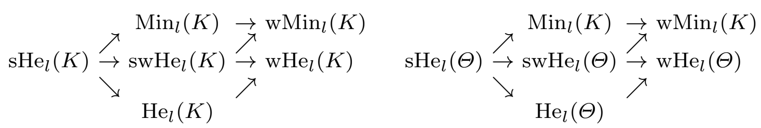

Fig. 1 Inclusion relationships. The arrow '→' means '⊂'. In this scheme we have omitted F and S. If F is K-compact valued, then $\mathrm{swHe}_l(K) = \mathrm{wHe}_l(K)$ and $\mathrm{swHe}_l(\Theta) = \mathrm{wHe}_l(\Theta)$.

In the next proposition, it is proved that two of the inclusions in (11) become equalities under a suitable assumption.

Proposition 3.3 *If the following condition is satisfied:*

$$\forall K' \in \mathcal{H}_K,\ \exists \eta \in (0,\delta) \text{ such that } K_\eta \subset K', \tag{22}$$

then $\mathrm{sHe}_l(F,S,K) = \mathrm{sHe}_l(F,S,\Theta)$ *and* $\mathrm{swHe}_l(F,S,K) = \mathrm{swHe}_l(F,S,\Theta)$*.*

Proof For the first equality, inclusion "$\supset$" is true by (11). To prove the reverse inclusion, let $x_0 \in \mathrm{sHe}_l(F,S,K)$. Then (8) holds for some $K' \in \mathcal{H}_K$, which means that $F(x_0) \not\subset F(x)+K'$ for all $x \in S\setminus \mathrm{E}_l(x_0)$. By assumption, $K_\eta \subset K'$ for some $\eta \in (0,\delta)$, and so, $F(x_0) \not\subset F(x)+K_\eta$ for all $x \in S \setminus \mathrm{E}_l(x_0)$, hence $x_0 \in \mathrm{sHe}_l(F,S,\Theta)$.

The second equality is similarly proved.

In general, $\mathrm{He}_l(F,S,K) \neq \mathrm{He}_l(F,S,\Theta)$, even if (22) holds, as the following example shows.

Example 3.2 Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$, $K := \mathbb{R}^2_+$, $\Theta := \{(y_1,y_2) \in \mathbb{R}^2_+ : y_1+y_2 = 1\}$, $S := [0,1]$ and $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ given by $F(x) := \{(y_1,y_2) : y_2 = x - y_1\}$. If we choose $K' := \{(y_1,y_2) : y_1+2y_2 \geq 0,\ y_1+4y_2 \geq 0\} \in \mathcal{H}_K$, one has $F(1) \subset F(x)+K'$ and $F(x) \subset F(1)+K'$ for all $x \in S$. Hence $1 \in \mathrm{Min}_l(F,S,K')$ and so $1 \in \mathrm{He}_l(F,S,K)$. However, $F(1) \subset F(0)+K_\eta$ and $F(0) \not\subset F(1)+K_\eta$ for all $\eta \in (0,\delta)$, and therefore $1 \notin \mathrm{Min}_l(F,S,K_\eta)$ for all $\eta \in (0,\delta)$. In consequence, $1 \notin \mathrm{He}_l(F,S,\Theta)$.

Besides, one has $1 \notin \mathrm{sHe}_l(F,S,K)$ since $F(1) \subset F(0)+K'$ for all $K' \in \mathcal{H}_K$, Thus $1 \in \mathrm{He}_l(F,S,K) \setminus \mathrm{sHe}_l(F,S,K)$, which shows that in general the first inclusion in (13) is strict, i.e., $\mathrm{sHe}_l(F,S,K) \neq \mathrm{He}_l(F,S,K)$.

Remark 3.3 (i) If Θ is a compact base of K, then condition (22) holds (see the proof of Proposition 2.4.6(iii) in [26]).

(ii) If Y is finite-dimensional, then for any pointed closed convex cone $K \subset Y$, there exists a compact base Θ of K (see, for instance, [10, page 3]), and consequently (22) is satisfied.

(iii) It is an open question when condition (22) is true in infinite-dimensional spaces.

Next, we study the relationships between $\leq^l_K$-Henig and the classical Henig notion, that is, the concepts of Henig proper efficiency with the set and the vector criterion. First, we assume that $F = f$ is single-valued (the function $f : X \to Y$ is considered as a map), and then, in Proposition 3.5, we consider a set-valued map F.

For a function f, parts (i)-(ii) and (iii)-(iv) in Definition 3.1 reduce, respectively, to the corresponding notion in parts (iii) and (iv) of Definition 2.2, according to the following proposition.

Proposition 3.4 *(i)* $\mathrm{He}_l(f,S,K) = \mathrm{sHe}_l(f,S,K) = \mathrm{He}(f,S,K)$*.*

(ii) $\mathrm{swHe}_l(f,S,\Theta) = \mathrm{wHe}_l(f,S,\Theta) = \mathrm{He}(f,S,\Theta)$*.*

(iii) If condition (22) *holds, then the six sets in parts (i) and (ii) are equal.*

Proof (i) Using (13), we have $\mathrm{sHe}_l(f,S,K)\subset \mathrm{He}_l(f,S,K)$, so if we prove the chain $\mathrm{He}_l(f,S,K)\subset \mathrm{He}(f,S,K)\subset \mathrm{sHe}_l(f,S,K)$, we will have finished.

Let $x_0\in \mathrm{He}_l(f,S,K)$. Then from the definition and in view of Remark 2.5, there exists $K'\in \mathcal{H}_K$ such that

$$x\in S,\ f(x)\leq_{K'} f(x_0) \ \Rightarrow\ f(x_0)\leq_{K'} f(x),$$

or equivalently,

$$x\in S,\ f(x)-f(x_0)\in -K' \ \Rightarrow\ f(x_0)-f(x)\in -K'.$$

As K' is pointed, it follows that

$$(f(S)-f(x_0))\cap(-K'\setminus\{0\})=\emptyset, \tag{23}$$

and consequently, $x_0\in \mathrm{Min}(f,S,K')$, which implies that $x_0\in \mathrm{He}(f,S,K)$.

Now, let $x_0\in \mathrm{He}(f,S,K)$. Then there exists $K'\in\mathcal{H}_K$ such that (23) holds. Let us observe that a point $x\in S$ satisfies $x\in \mathrm{E}_l(x_0)$ if and only if $f(x)\leq_K f(x_0)$ and $f(x_0)\leq_K f(x)$, equivalently, if and only if $f(x)-f(x_0)\in(-K)\cap K=\{0\}$ since K is pointed. Therefore,

$$\mathrm{E}_l(x_0)=\{x\in S:\ f(x)=f(x_0)\}.$$

Let us see that $f(x)\nleq_{K'} f(x_0)$ for all $x\in S\setminus \mathrm{E}_l(x_0)$. By contradiction, assume that $f(x_1)-f(x_0)\in -K'$ for some $x_1\in S\setminus \mathrm{E}_l(x_0)$. As $x_1\notin \mathrm{E}_l(x_0)$, one has $f(x_1)\neq f(x_0)$. So $f(x_1)-f(x_0)\in -K'\setminus\{0\}$, which contradicts (23). In consequence, $x_0\in \mathrm{sHe}_l(f,S,K)$.

(ii) The following characterization is true for a function:

$$x_0\in \mathrm{swHe}_l(f,S,\Theta)\Leftrightarrow \exists\eta\in(0,\delta): f(x)\not\ll_{K_\eta} f(x_0)\quad \forall x\in S. \tag{24}$$

Indeed, since f is K-compact valued, by Proposition 3.2(i) we have that $x_0\in \mathrm{swHe}_l(f,S,\Theta)$ if and only if there exists $\eta\in(0,\delta)$ such that $x_0\in \mathrm{wMin}_l(f,S,K_\eta)$. By Lemma 3.2(i), this last condition is equivalent to $f(x)\not\ll_{K_\eta} f(x_0)$ for all $x\in S$, and consequently (24) is proved.

Now, condition (24) is equivalent to say that $(f(S)-f(x_0))\cap(-\operatorname{int}K_\eta)=\emptyset$, which is equivalent by definition to say that $f(x_0)\in \mathrm{wMin}(f(S),K_\eta)$. This implies by Proposition 2.1 that $f(x_0)\in \mathrm{He}(f(S),\Theta)$, and therefore, by definition, $x_0\in \mathrm{He}(f,S,\Theta)$. Finally, the implication "$x_0\in \mathrm{He}(f,S,\Theta)\Rightarrow x_0\in \mathrm{swHe}_l(f,S,\Theta)$" follows easily from the above argument. It follows from Proposition 3.2(i) that $\mathrm{swHe}_l(f,S,\Theta)=\mathrm{wHe}_l(f,S,\Theta)$.

(iii) It suffices to prove that $\mathrm{He}(f,S,\Theta)=\mathrm{He}(f,S,K)$, and this is equivalent to prove that $\mathrm{He}(A,\Theta)=\mathrm{He}(A,K)$ with $A=f(S)$. In view of (4) we only need to check that $\mathrm{He}(A,K)\subset \mathrm{He}(A,\Theta)$. But this is immediate, since if $a_0\in \mathrm{He}(A,K)$ then there exists $K'\in\mathcal{H}_K$ such that $(A-a_0)\cap(-K'\setminus\{0\})=\emptyset$. From (22), we have $K_\eta\subset K'$ for some $\eta\in(0,\delta)$, and so $(A-a_0)\cap(-K_\eta\setminus\{0\})=\emptyset$. Therefore $a_0\in \mathrm{He}(A,\Theta)$.

Proposition 3.5 *Let* $x_0 \in S$.

(i) If there exists a point $y_0 \in F(x_0) \setminus F(x)$ *for all* $x \in S \setminus \mathrm{E}_l(x_0)$ *and* $y_0 \in \mathrm{He}(F(S), K)$*, then* $x_0 \in \mathrm{sHe}_l(F, S, K)$.

(ii) If $x_0 \in \mathrm{He}(F, S, \Theta)$*, then* $x_0 \in \mathrm{swHe}_l(F, S, \Theta)$.

Proof (i) By hypothesis there exists $K' \in \mathcal{H}_K$ such that $(F(S) - y_0) \cap (-K' \setminus \{0\}) = \emptyset$, i.e.,

$$y - y_0 \notin -K' \setminus \{0\}, \quad \forall y \in F(x),\ \forall x \in S. \tag{25}$$

Let us prove $F(x) \not\leq^l_{K'} F(x_0)$ for all $x \in S \setminus \mathrm{E}_l(x_0)$. By contradiction, assume that there exists $x_1 \in S \setminus \mathrm{E}_l(x_0)$ satisfying $F(x_1) \leq^l_{K'} F(x_0)$, so $F(x_0) \subset F(x_1) + K'$. As $y_0 \in F(x_0) \setminus F(x_1)$ by assumption, one has $y_0 = y_1 + k'$ for some $y_1 \in F(x_1)$ and $k' \in K' \setminus \{0\}$, which contradicts (25).

(ii) Let $x_0 \in \mathrm{He}(F, S, \Theta)$, then by using (6), there exist $y_0 \in F(x_0)$ and $\eta \in (0, \delta)$ such that $y_0 \in \mathrm{wMin}(F(S), K_\eta)$, i.e.,

$$(F(x) - y_0) \cap (-\operatorname{int} K_\eta) = \emptyset, \quad \forall x \in S. \tag{26}$$

Let us see that

$$F(x) \not\ll^l_{K_\eta} F(x_0), \quad \forall x \in S,$$

which implies that (9) holds and so part (ii) is proved. By contradiction, assume that there is $x_1 \in S$ such that $F(x_1) \ll^l_{K_\eta} F(x_0)$. Then $F(x_0) \subset F(x_1) + \operatorname{int} K_\eta$. As $y_0 \in F(x_0)$, there exists $y_1 \in F(x_1)$ satisfying $y_0 \in y_1 + \operatorname{int} K_\eta$, which contradicts (26).

Part (ii) shows that the notion of strict weak $\leq^l_K$-Henig proper solution is weaker than the notion of vector Henig proper solution.

The previous proposition is very useful to find $\leq^l_K$-Henig proper solutions. We illustrate it with an example.

Example 3.3 Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$, $K := \mathbb{R}^2_+$, $S := \mathbb{R}$ and $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined by $F(t) := f(t) + B$, where $f(t) := (t, t^2)$ and $B := \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 1\}$. To visualize this example, see Fig. 2.

For each $t \in \mathbb{R}$, the tangent line to the parabola $y_2 = y_1^2$ (image of f) at the point $f(t)$ is $2ty_1 - y_2 - t^2 = 0$, and the unit normal vector is $\frac{1}{\sqrt{4t^2+1}}(2t, -1)$. So the point $y_t = f(t) + \frac{1}{\sqrt{4t^2+1}}(2t, -1) \in F(t)$. Moreover, $y_t \notin F(x)$ for all $x \neq t$ since $\|y_t - f(x)\| > 1$. Note that $\mathrm{E}_l(t) = \{t\}$.

For each $t < 0$ we have $y_t \in \mathrm{He}(F(S), K)$. Indeed, the slope of the tangent line to the curve $t \mapsto y_t$ is $2t$. So, choosing $K'_t := \{(y_1, y_2) : y_2 \geq 4ty_1,\ y_2 \geq y_1/(4t)\} \in \mathcal{H}_K$ one has $(F(S) - y_t) \cap (-K'_t \setminus \{0\}) = \emptyset$, which means that $y_t \in \mathrm{Min}(F(S), K'_t)$, and therefore, $y_t \in \mathrm{He}(F(S), K)$. Now, by Proposition 3.5(i) it follows that $t \in \mathrm{sHe}_l(F, S, K)$ for all $t < 0$.

The following example shows that the reciprocal implication of Proposition 3.5(ii) is not true in general.

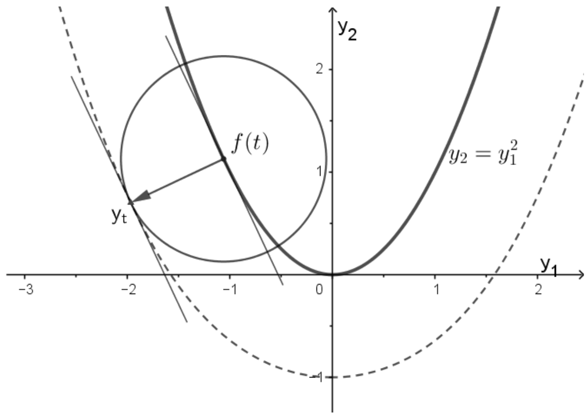

Fig. 2 Illustration of Example 3.3

Example 3.4 Let $X := \mathbb{R}$, $S := \{0,1,2\}$, $Y := \mathbb{R}^2$, $K := \mathbb{R}^2_+$, $\Theta := \{(y_1,y_2) \in \mathbb{R}^2_+ : y_1 + y_2 = 1\}$, $F : S \rightrightarrows \mathbb{R}^2$ given by $F(0) := \{(3,0),(0,3)\}$, $F(1) := \{(1,-1)\}$ and $F(2) := \{(-1,1)\}$. It is easy to check that $0 \in \mathrm{swHe}_l(F,S,\Theta)$ but $0 \notin \mathrm{He}(F,S,\Theta)$. This example also shows that Proposition 3.4(ii) is not true for a set-valued map F instead of f.

This example also shows that Proposition 3.4(ii) is not true for a set-valued map F instead of f.

Remark 3.4 After studying the main relations between the different notions of $\leq^l_K$-Henig proper solutions and minimality, we have seen that strict $\leq^l_K$-Henig proper solutions are minimal solutions and strict weak $\leq^l_K$-Henig proper solutions are weak minimal solutions (Proposition 3.1). This latter is also valid for $\leq^l_K$-Henig (and weak $\leq^l_K$-Henig) proper solutions when F is K-compact valued (see Proposition 3.2 and Fig. 1). We have also stated the $\leq^l_K$-Henig proper solution generalizes well the vector notion of Henig proper solution when we consider a function (Proposition 3.4). However, in general, the inclusion $\mathrm{He}(F,S,K) \subset \mathrm{Min}_l(F,S,K)$ is not true. If we want to get this inclusion we have the possibility to give more restrictive notions of $\leq^l_K$-Henig proper solutions:

(i) The first one follows the idea of other proper notions as f.e., Geoffrion proper notion, which requires minimality and a specific property. In consequence, we can define

$$\mathrm{He2}_l(F,S,K) := \mathrm{He}_l(F,S,K) \cap \mathrm{Min}_l(F,S,K).$$

(ii) The second would be given as follows: $x_0 \in \mathrm{He3}_l(F,S,K)$ if and only if there exists $K' \in \mathcal{H}_K$ such that $x \in S$, $F(x) \leq^l_{K'} F(x_0)$ implies $F(x_0) \leq^l_K F(x)$.

In this paper we do not study these notions because they are very similar to the study done for the notions in Definition 3.1 and Remark 3.1.

To finish this section, we are going to state a characterization of weak $\leq^l_K$-Henig proper solutions of (SOP) through scalarization, which will be used in the next section. Previously, we show a necessary result to achieve our purpose, which can be seen in [24, Theorem 5.7]. Given a scalar function $\phi : X \to \mathbb{R}$, we denote $\operatorname{argmin}_{x\in S} \phi(x) := \{x_0 \in S : \phi(x) \geq \phi(x_0) \ \forall x \in S\}$.

Theorem 3.1 *Suppose that K is solid, $x_0 \in S$ and $F(x_0)$ is K-compact. Then $x_0 \in \mathrm{wMin}_l(F, S, K)$ if and only if $x_0 \in \operatorname{argmin}_{x\in S} \mathbb{D}^{si}_K(F(x), F(x_0))$.*

In the following result we state a characterization for weak $\leq^l_K$-Henig proper solutions wrt a base.

Theorem 3.2 *Assume that $x_0 \in S$ and $F(x_0)$ is K-compact. Then the following statements are equivalent:*
(i) $x_0 \in \mathrm{wHe}_l(F, S, \Theta)$.
(ii) There exists $\eta \in (0, \delta)$ such that $x_0 \in \operatorname{argmin}_{x\in S} \mathbb{D}^{si}_{K_\eta}(F(x), F(x_0))$.

Proof By Definition 3.1(iii), $x_0 \in \mathrm{wHe}_l(F, S, \Theta)$ if and only if there exists $\eta \in (0, \delta)$ such that $x_0 \in \mathrm{wMin}_l(F, S, K_\eta)$. By Lemma 3.1 we have that $F(x_0)$ is K_η-compact. Moreover, K_η is solid by Lemma 2.1. Now, the equivalence between (i) and (ii) is obtained by applying Theorem 3.1.

Let us note that this theorem does not require the cone K to be solid.

4 Lagrange multipliers for Henig proper solutions of (SOP)

In this section, from a Lagrange multiplier rule for weak $\leq^l_K$-minimal solutions of a set optimization problem and the results of the previous section, we obtain a characterization for Henig proper solutions with the set criterion of a convex set optimization problem with constraints when the decision space is finite dimensional.

First, we recall some necessary notions and properties.

Definition 4.1 A set-valued map $F : X \rightrightarrows Y$ is said to be K-convex (resp., convex) if, for all $x_1, x_2 \in X$ and for all $\lambda \in [0, 1]$, one has

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + K$$

$$(\text{resp., } \lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) \ ).$$

Remark 4.1 (i) F is convex (resp., K-convex) if and only if $\operatorname{gr} F$ (resp., $\operatorname{epi} F$) is convex ([16, Proposition 2.3]).

(ii) It is clear that if F is convex, then F is K-convex, but the converse is not true. Indeed, the map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as $F(x) = \{x^2\}$ is K-convex, for $K = \mathbb{R}_+$, since $\operatorname{epi} F = \{(x, y) : x \in \mathbb{R}, y \geq x^2\}$ is a convex set, but F is not convex, since $\operatorname{gr} F = \{(x, x^2) : x \in \mathbb{R}\}$ is not a convex set.

(iii) F is K-convex if and only if F_K is convex.

Definition 4.2 ([10, Definition 2.5.16)]) Suppose that X is a normed space. A set-valued map $F : X \rightrightarrows Y$ is said to be K-Hausdorff upper continuous (K-H-u.c.) at x_0 if for any neighbourhood U of $0 \in Y$, there exists a neighbourhood V of x_0 such that $F(x) \subset F(x_0) + U + K$ for all $x \in V$.

We say that F is K-H-u.c. if F is K-H-u.c. at each point in X.

For a single-valued map $F = f$, it is said that $f : X \to Y$ is K-continuous at x_0 ([32, Chapter 1, Definition 5.1]) if f is K-H-u.c. at x_0.

Let X, Y and Z be normed spaces, $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ be two set-valued maps, $K \subset Y$ and $C \subset Z$ be closed convex cones.

We deal with the following set optimization problem

$$\text{Minimize } F(x) \quad \text{subject to } x \in S_G \cap Q, \tag{CSOP}$$

where Q is a nonempty convex subset of X and

$$S_G := \{x \in X : G(x) \cap (-C) \neq \emptyset\}.$$

Let $A \subset X$ be a nonempty convex set and $a \in A$. The normal cone to A at a is given by $N(A, a) := \{x^* \in X^* : \langle x^*, x - a\rangle \leq 0, \forall x \in A\}$.

Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ and $x_0 \in \operatorname{dom} \varphi := \{x \in X : \varphi(x) < +\infty\}$. We remind that the subdifferential of φ at x_0 in the sense of Convex Analysis is given by

$$\partial\varphi(x_0) := \{x^* \in X^* : \varphi(x) \geq \varphi(x_0) + \langle x^*, x - x_0\rangle, \forall x \in X\}.$$

When $F : X \rightrightarrows Y$ is convex, the coderivative of F at $(x, y) \in \operatorname{gr} F$ (see [1]) is the set-valued map $D^*F : Y^* \rightrightarrows X^*$ given by

$$D^*F(x, y)(y^*) = \{x^* \in X^* : (x^*, -y^*) \in N(\operatorname{gr} F, (x, y))\}.$$

In this part, as an application of the previous parts, we will state a characterization for weak $\leq^l_K$-Henig proper solutions wrt a base with the set criterion of solution. For this aim, we will use the following characterization for weak minimal solutions of (CSOP), which has been proved in [19].

Theorem 4.1 *[19, Theorem 6] Let $x_0 \in S_G \cap Q$. Assume that $X = \mathbb{R}^n$, $Q \subset X$ is a convex set and K and C are solid. Suppose the following:*

(i) $F(x_0)$ is compact and F is K-H-u.c., K-convex, K-proper valued and K-closed valued,

(ii) $G(x_0)$ is compact and G is C-compact valued, C-H-u.c. and C-convex,

(iii) the Slater constraint qualification holds, i.e.,

$$\text{there exists } x_1 \in Q \text{ such that } G(x_1) \cap (-\operatorname{int} C) \neq \emptyset. \tag{27}$$

Then x_0 is a weak $\leq^l_K$-minimal solution of (CSOP) if and only if there exist $\mu \geq 0$, $r \leq n + 1$, $b_i \in \operatorname{wMin}(F(x_0), K)$, $y_i^* \in K^+ \setminus \{0\}$, $i = 1, \ldots, r$, $z_0 \in G(x_0) \cap (-C)$ and $z^* \in C^+$ such that $\langle z^*, z_0\rangle = 0$,

$$0 \in \sum_{i=1}^{r} D^*(F + K)(x_0, b_i)(y_i^*) + \mu D^*(G + C)(x_0, z_0)(z^*) + N(Q, x_0),$$

and $\mu = 0$ whenever $G(x_0) \cap (-\operatorname{int} C) \neq \emptyset$.

Remark 4.2 Actually, [19, Theorem 6] is proved for F H-u.c. instead of K-H-u.c., and G H-u.c. instead of C-H-u.c. But Theorem 6 can be proved with the weaker hypotheses. Indeed, in the proof of Theorem 6 the hypotheses F is H-u.c. and G is H-u.c. are only used when it is applied [19, Lemma 6] to prove that $F+K$ and $G+C$ are closed, but now, when F is K-H-u.c. and G is C-H-u.c., we can use [20, Lemma 5] and we obtain the same conclusion: $F+K$ and $G+C$ are closed.

Lemma 4.1 *Let $F : X \rightrightarrows Y$ be K-convex and let $\tilde{K}$ be a closed convex cone with $K \subset \tilde{K}$. Then*
(i) F is $\tilde{K}$-convex.
(ii) If, in addition, $y_0 \in F(x_0)$ and $y^* \in \tilde{K}^+$, then

$$D^*(F+K)(x_0, y_0)(y^*) = D^*(F+\tilde{K})(x_0, y_0)(y^*).$$

Proof (i) As F is K-convex, we have

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + K \subset F(\lambda x_1 + (1-\lambda)x_2) + \tilde{K}$$

for all $x_1, x_2 \in X$, $\lambda \in (0,1)$, which implies that F is $\tilde{K}$-convex.

(ii) First let us observe that $F+K$ and $F+\tilde{K}$ are convex by Remark 4.1(iii) and so the coderivatives in part (ii) have sense.

Let $x^* \in X^*$ with $x^* \in D^*(F+K)(x_0, y_0)(y^*)$. Then by definition $(x^*, -y^*) \in N(\mathrm{gr}(F+K), (x_0, y_0))$, which means that $\langle (x^*, -y^*), (x-x_0, y+k-y_0)\rangle \le 0$ for all $x \in X, y \in F(x), k \in K$, which is equivalent to

$$\langle y^*, y+k-y_0\rangle \ge \langle x^*, x-x_0\rangle, \quad \forall x \in X, y \in F(x), k \in K.$$

In particular, taking $k=0$, $\langle y^*, y-y_0\rangle \ge \langle x^*, x-x_0\rangle$ for all $x \in X, y \in F(x)$. As $y^* \in \tilde{K}^+$, we have $\langle y^*, \tilde{k}\rangle \ge 0$ for all $\tilde{k} \in \tilde{K}$. Adding the last two inequalities it results

$$\langle y^*, y+\tilde{k}-y_0\rangle \ge \langle x^*, x-x_0\rangle, \quad \forall x \in X, y \in F(x), \tilde{k} \in \tilde{K},$$

which means that $(x^*, -y^*) \in N(\mathrm{gr}(F+\tilde{K}), (x_0, y_0))$. From here we conclude that $x^* \in D^*(F+\tilde{K})(x_0, y_0)(y^*)$, and so $D^*(F+K)(x_0, y_0)(y^*) \subset D^*(F+\tilde{K})(x_0, y_0)(y^*)$.

The reverse inclusion, $D^*(F+\tilde{K})(x_0, y_0)(y^*) \subset D^*(F+K)(x_0, y_0)(y^*)$, follows from the fact that $N(\mathrm{gr}(F+\tilde{K}), (x_0, y_0)) \subset N(\mathrm{gr}(F+K), (x_0, y_0))$ since $\mathrm{gr}(F+K) \subset \mathrm{gr}(F+\tilde{K})$.

In the next theorem, we establish a characterization for weak $\le_K^l$-Henig proper solutions to the constrained problem (CSOP) with the set criterion.

Theorem 4.2 *Let $x_0 \in S := S_G \cap Q$, and assume that Θ is a closed base of K and C is solid. Suppose the following:*
(i) $X = \mathbb{R}^n$,
(ii) $F(x_0)$ is compact and F is K-compact valued, K-H-u.c. and K-convex,
(iii) $G(x_0)$ is compact and G is C-compact valued, C-H-u.c. and C-convex,

(iv) Q is a convex set,

(v) the Slater constraint qualification (27) *holds.*

Then $x_0 \in \mathrm{wHe}_l(F, S, \Theta)$ if and only if there exist $r \leq n+1$, $\mu \geq 0$, $z_0 \in G(x_0) \cap (-C)$ and $z^* \in C^+$, and for each $i = 1, \ldots, r$, there exist

$$b_i \in \mathrm{He}(F(x_0), \Theta) \tag{28}$$

and $y_i^* \in K^{\Delta}(\Theta)$ such that $\langle z^*, z_0 \rangle = 0$,

$$0 \in \sum_{i=1}^{r} D^*(F+K)(x_0, b_i)(y_i^*) + \mu D^*(G+C)(x_0, z_0)(z^*) + N(Q, x_0), \tag{29}$$

and $\mu = 0$ whenever $G(x_0) \cap (-\operatorname{int} C) \neq \emptyset$.

Proof ($\Rightarrow$) First, by Definition 3.1(iii), $x_0 \in \mathrm{wHe}_l(F, S, \Theta)$ if and only if there exists $\eta \in (0, \delta)$ such that $x_0 \in \mathrm{wMin}_l(F, S, K_\eta)$.

Second, we are going to apply Theorem 4.1 to the pair (F, K_η). Let us check all its assumptions: 1) K_η is a solid, pointed, closed convex cone by Lemma 2.1(i). 2) F is K_η-compact valued because by hypothesis $F(x)$ is K-compact for all $x \in X$ and so, by Lemma 3.1, $F(x)$ is K_η-compact. Therefore, $F(x)$ is K_η-closed and K_η-bounded, and so it is also K_η-proper (see Remark 2.1). In consequence, F is K_η-closed and K_η-proper valued. 3) F is K_η-convex by Lemma 4.1(i) since F is K-convex by hypothesis. 4) The assumption that F is K_η-H-u.c. can be easily checked since F is K-H-u.c.

In consequence, there exist $r \leq n+1$, $\mu \geq 0$, $z_0 \in G(x_0) \cap (-C)$ and $z^* \in C^+$, and for each $i = 1, \ldots, r$, there exist

$$b_i \in \mathrm{wMin}(F(x_0), K_\eta) \tag{30}$$

and $y_i^* \in K_\eta^+ \setminus \{0\}$ such that $\langle z^*, z_0 \rangle = 0$,

$$0 \in \sum_{i=1}^{r} D^*(F+K_\eta)(x_0, b_i)(y_i^*) + \mu D^*(G+C)(x_0, z_0)(z^*) + N(Q, x_0), \tag{31}$$

and $\mu = 0$ if $G(x_0) \cap (-\operatorname{int} C) \neq \emptyset$.

In view of (30), from Proposition 2.1 it follows that (28) holds.

As $y_i^* \in K_\eta^+ \setminus \{0\}$, by Lemma 2.2(i) we have $y_i^* \in K^{\Delta}(\Theta)$.

As $K \subset K_\eta$, in view of Lemma 4.1 we deduce that $D^*(F+K_\eta)(x_0, b)(y^*) = D^*(F+K)(x_0, b)(y^*)$ for all $y^* \in K_\eta^+$ and all $b \in F(x_0)$. In consequence, from (31) it follows (29).

($\Leftarrow$) Suppose that there exist $\mu \geq 0$, $z_0 \in G(x_0) \cap (-C)$ and $z^* \in C^+$, $r \leq n+1$, $y_i^* \in K^{\Delta}(\Theta)$, b_i, $i = 1, \ldots, r$ satisfying $\langle z^*, z_0 \rangle = 0$, (28)-(29) and $\mu = 0$ whenever $G(x_0) \cap (-\operatorname{int} C) \neq \emptyset$. By Proposition 2.1 it follows that there exist $\eta_i \in (0, \delta)$, $i = 1, \ldots, r$ such that $b_i \in \mathrm{wMin}(F(x_0), K_{\eta_i})$. Let $\eta_0 := \min_{i=1,\ldots,r} \eta_i$, then $b_i \in \mathrm{wMin}(F(x_0), K_{\eta_0})$ for all $i = 1, \ldots, r$ since $K_{\eta_0} \subset K_{\eta_i}$.

As $y_i^* \in K^{\Delta}(\Theta)$, by Lemma 2.2(i) we derive that for each $i = 1, \dots, r$, there exists $\bar{\eta}_i \in (0, \delta)$ with $y_i^* \in K^+_{\bar{\eta}_i} \backslash \{0\}$. Let $\bar{\eta}_0 := \min_{i=1,\dots,r} \bar{\eta}_i$, then $y_i^* \in K^+_{\bar{\eta}_0} \backslash \{0\}$ for all $i = 1, \dots, r$.

Define $\eta := \min\{\eta_0, \bar{\eta}_0\}$. Then $b_i \in \mathrm{wMin}(F(x_0), K_\eta)$ and $y_i^* \in K^+_\eta \backslash \{0\}$ for all $i = 1, \dots, r$. Using Lemma 4.1(ii) we derive that $D^*(F + K)(x_0, b_i)(y_i^*) = D^*(F + K_\eta)(x_0, b_i)(y_i^*)$ and then, from (29) we deduce that (31) holds. Thus we can apply the reciprocal part of Theorem 4.1 to the pair (F, K_η) getting that $x_0 \in \mathrm{wMin}_l(F, S, K_\eta)$. Now, by Definition 3.1(iii) the required conclusion is obtained.

Finally, as an application of Theorem 4.2 we state a characterization of Henig proper solutions for a vector optimization problem, i.e., when $F = f$ and $G = g$ are single-valued and cone-convex.

Given $f : X \to Y$, $g : X \to Z$ and $Q \subset X$, let $S_g := \{x \in X : g(x) \in -C\}$. We consider the vector optimization problem

$$\text{(VP) Minimize } f(x) \quad \text{subject to} \quad x \in S_g \cap Q.$$

Corollary 4.1 *Assume that* $X = \mathbb{R}^n$, $x_0 \in S_g \cap Q$, $f : X \to Y$ *is* K*-convex and* K*-continuous,* $g : X \to Z$ *is* C*-convex and* C*-continuous,* Q *is convex,* Θ *is a closed base of* K *and the Slater constraint qualification holds, i.e.,*

$$\textit{there exists } x_1 \in Q : \; g(x_1) \in -\operatorname{int} C.$$

Then $x_0 \in \mathrm{He}(f, S_g \cap Q, \Theta)$ *if and only if there exist* $\mu \geq 0$, $y^* \in K^{\Delta}(\Theta)$, $z^* \in C^+$ *such that* $\langle z^*, g(x_0)\rangle = 0$,

$$0 \in \partial(y^* \circ f)(x_0) + \mu \partial(z^* \circ g)(x_0) + N(Q, x_0),$$

and $\mu = 0$ *whenever* $g(x_0) \in -\operatorname{int} C$.

Proof Assume that $x_0 \in \mathrm{He}(f, S_g \cap Q, \Theta)$. Then by Proposition 3.4(ii) it follows that $x_0 \in \mathrm{wHe}_l(f, S_g \cap Q, \Theta)$. Now, the conclusion is obtained from Theorem 4.2 with $F = f$ and $G = g$ taking into account the following facts:

(i) $D^*(f+K)(x_0)(y^*) = \partial(y^* \circ f)(x_0)$ and $D^*(g+C)(x_0)(z^*) = \partial(z^* \circ g)(x_0)$ as it can be easily checked from the definitions. Let us note that $y^* \circ f$ and $z^* \circ g$ are convex functions.

(ii) If $x_i^* \in \partial(y_i^* \circ f)(x_0)$ and $y_i^* \in K^{\Delta}(\Theta)$ for $i = 1, \dots, r$, then $\sum_{i=1}^r x_i^* \in \partial(y^* \circ f)(x_0)$, where $y^* = \sum_{i=1}^r y_i^* \in K^{\Delta}(\Theta)$. This fact can also be easily verified.

The reverse implication follows also from Theorem 4.2.

Remark 4.3 We may obtain the same result considering a general normed space X instead of $X = \mathbb{R}^n$. For this, we apply first Definition 2.2(iv) and then Corollary 1 in [19] following the same ideas that in the proof of Theorem 4.2 and taking into account Remark 4.2.

We illustrate some of the main results in the paper in the following example.

Example 4.1 With the data of Example 3.3 with $Q := \mathbb{R}$ and without constraint G, $K := \mathbb{R}^2_+$ and $\Theta := \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 + y_2 = 1\}$. It is clear that F is compact-valued and K-H-u.c.. Moreover, F is K-convex by Proposition 3.5(c) in [16] and the Slater constraint qualification is not necessary. Condition (29) at a point $t \in \mathbb{R}$ and $b \in F(t)$ with $r = 1$ becomes

$$0 \in D^* F_K(t, b)(y^*). \tag{32}$$

Let us check if it is satisfied with $b \in \mathrm{He}(F(t), \Theta)$ and $y^* = (y_1^*, y_2^*) \in K^{\Delta}(\Theta) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 > 0,\, y_2 > 0\}$. One has $\mathrm{gr}\, F_K = \{(t, y_1, y_2) \in \mathbb{R}^3 : (y_1, y_2) \in F(t) + \mathbb{R}^2_+\}$ and $F(t) = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - t)^2 + (y_2 - t^2)^2 \le 1\}$. Therefore, $F(t) + \mathbb{R}^2_+ = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 = t^2 - \sqrt{1 - (y_1 - t)^2},\, t - 1 \le y_1 \le t\} + \mathbb{R}^2_+$. It is well-known that $b = (b_1, b_2) \in \mathrm{He}(F(t), \Theta)$ if and only if $b_2 = t^2 - \sqrt{1 - (b_1 - t)^2}$ and $t - 1 < b_1 < t$. Consider the parametric surface $z = z(t, v) = (z_1, z_2, z_3)$, which is contained in $\mathrm{bd}\,\mathrm{gr}\, F_K$, given by

$$z_1 = t,\ z_2 = v,\ z_3 = t^2 - \sqrt{1 - (v - t)^2} \ \text{ with } t \in \mathbb{R},\ t - 1 < v < t. \tag{33}$$

A normal vector to this surface at the point $z(t, v)$ is

$$\begin{aligned} w(t, v) &:= \frac{\partial z}{\partial v} \times \frac{\partial z}{\partial t} = (0, 1, (v - t)/A) \times (1, 0, 2t - (v - t)/A) \\ &= (2tA - (v - t), v - t, -A)/A, \end{aligned}$$

where $A := \sqrt{1 - (v - t)^2}$. One has $N(\mathrm{gr}\, F_K, z(t, v)) = \{\alpha w(t, v) : \alpha \ge 0\}$. Condition (32) is satisfied if and only if $(0, -y_1^*, -y_2^*) = \alpha w(t, v)$ for some $\alpha > 0$ since $y^* \ne 0$. Choosing $\alpha := A$, we derive

$$2t\sqrt{1 - (v - t)^2} - (v - t) = 0,\ \ -y_1^* = v - t,\ \ -y_2^* = -\sqrt{1 - (v - t)^2}.$$

This system has no solution for $t \ge 0$ because the first equation is incompatible since $v - t < 0$ in view of (33). For $t < 0$, it has the solution $v = v_t := t + \frac{2t}{\sqrt{1+4t^2}}$, $y_1^* = \frac{-2t}{\sqrt{1+4t^2}}$, $y_2^* = \frac{1}{\sqrt{1+4t^2}}$. For $t < 0$, one has that $y^* \in K^{\Delta}(\Theta)$, and moreover, $b = b_t := (t + 2t/\sqrt{1 + 4t^2}, t^2 - 1/\sqrt{1 + 4t^2}) \in \mathrm{He}(F(t), \Theta)$ (note that b_t is the same point considered in Example 3.3). Therefore, by Theorem 4.2 all the points $t < 0$ are weak $\le^l_K$-Henig proper solutions. The above argument does not allow to assure that $t \notin \mathrm{wHe}_l(F, \mathbb{R}, \Theta)$ for $t \ge 0$ because we have used $r = 1$.

For $t > 0$ it is easy to check that $t \notin \mathrm{wMin}_l(F, \mathbb{R}, K)$, and as $\mathrm{wHe}_l(F, \mathbb{R}, \Theta) = \mathrm{swHe}_l(F, \mathbb{R}, \Theta)$ by Proposition 3.2(i) since F is K-compact valued, we derive that $t \notin \mathrm{wHe}_l(F, \mathbb{R}, \Theta)$ (see (15)). Finally, by definition we have that $t = 0 \notin \mathrm{wHe}_l(F, \mathbb{R}, \Theta)$ since for all $\eta \in (0, \sqrt{2}/2)$ one has that $0 \notin \mathrm{wMin}_l(F, \mathbb{R}, K_\eta)$ as can be checked. Therefore, $\mathrm{wHe}_l(F, \mathbb{R}, \Theta) = (-\infty, 0)$.

5 Conclusions

In this paper, we have considered a set optimization problem for which we have defined new concepts of proper efficiency in the sense of Henig with the set criterion of solution, by taking into account preference relations between sets related to the order structure in the final space. The relations between these concepts and their counterparts defined by using the vector criterion of solution have been studied, as well as some of the main properties of these solutions. In Remark 3.4, we have presented other possibilities to define Henig proper solutions with the set criterion. So, this interesting topic deserves more attention and some research should be carried out in order to clarify which notion is more suitable and to study new properties.

Finally, a Lagrange multiplier rule for Henig proper solutions of a set optimization problem with a cone constraint has been obtained, by using the oriented distance as scalarizing functional, and considering convexity hypotheses. As an application, in the particular case when the set optimization problem reduces to a vector one, we have characterized Henig proper solutions of a constrained vector optimization problem.

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