Continuity of a scalarization in vector optimization with variable ordering structures and application to convergence of minimal solutions

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ABSTRACT
We consider a scalarization function, which was introduced by Eichfelder [1], based on the oriented distance of Hiriart-Urruty with respect to a general variable ordering structure. We first study the continuity of the composition of a set-valued map with the oriented distance. Then, using the obtained results, we study the continuity of the scalarization function by extending some concepts of continuity for cone-valued maps. As an application, convergence in the sense of Painlevé-Kuratowski of sets of weak minimal solutions is provided, with the vector criterion and a variable ordering structure. Illustrative examples are also given.

KEYWORDS
Scalarization in optimization; Continuity; Variable ordering structure; Oriented distance; Painlevé-Kuratowski convergence

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1. Introduction

The study of vector optimization problems with respect to (in short, w.r.t.) a variable ordering structure (in short, VOS) started some years ago and have recently gained interest [1–15] due to its applications, for instance, in medical image registration.

In a classical vector optimization problem (in short, VOP) [16,17], the values of the objective function are compared by means of a fixed ordering $\leq_K$ induced by a convex cone K, since the preferences are fixed along the problem. If the variability of the preferences throughout the problem is admitted [1,7,13] then a VOS arises, which was introduced by Yu [18] in 1974 through domination structures represented by a cone-valued map $\mathcal{K} : Y \rightrightarrows Y$, called ordering map, and generalized by Bergstresser et al. [19] in 1976.

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Variable ordering structures have been applied in different topics as, for example, in vector variational inequalities and complementarity problems [5,6], vector quasi-equilibrium problems [6,20], intensity-modulated radiation therapy [1,9], or to model preferences in multiobjective optimization problems [8,12].

By using an ordering map $\mathcal{K}$ not necessarily cone-valued, two binary relations have been studied [2,5,7,8,10,11,15] which allow introducing minimal and nondominated elements [2,4,5,7,8,18,19]. In a VOP w.r.t. a VOS, other optimality concepts were defined, namely, weak and strong solutions [5,10], approximate solutions [15], proper solutions [1,9,11], etc.

On the other hand, scalarization techniques are used to replace a vector or set optimization problem by a scalar optimization problem in order to facilitate the computation of the solutions of the original problem. Under a VOS, scalarization methods were used to characterize solutions in a VOP [1,7,9–11,15] and efficient elements in multiobjective optimization and variational inequalities [5,14], in optimality conditions of Fermat and Lagrange type [10], descent method to optimization [3], Ekeland's variational principle [15], numerical procedures [1,3,8], etc.

Gerstewitz's function [4,21] and oriented distance function of Hiriart-Urruty [22,23] are two important scalarization functions. In a VOP with a VOS, some extensions of Gerstewitz's function [4–6,10,15] have been given; however, a few extensions of the oriented distance [1,24,25] have been studied.

In the literature, properties about upper or lower semicontinuity for some extensions of Gerstewitz's function w.r.t. VOS, under suitable conditions, have been discussed in [4, Theorem 1.57], [5, Theorem 2.1] and [6, Theorem 2.1]. To the best of our knowledge, no type of continuity has been studied for the oriented distance w.r.t. a VOS.

Set convergence deals with limits of sequences of sets. It was introduced by Painlevé in 1902 and popularized by Hausdorff in 1927 and Kuratowski in 1933. Painlevé-Kuratowski convergence [26,27] is carried out through the notions of upper and lower limits of a sequence of sets, and these limits have been used in notions of semicontinuity, in different types of tangent cones, differentiation, approximation, stability to set optimization problems, etc.

Convergence of optimization problems under perturbations of the data is an important topic in optimization theory and methodology. Continuous behaviour of minimal point sets under perturbations is a central problem in stability and sensitivity analysis of a VOP. The upper and lower semicontinuity of minimal points arise in investigation of some other problems, for instance, in vector variational inequalities, duality theory, etc. There are many papers dealing with this topic under different assumptions (see, [28–30] and references therein), but all of them with a fixed ordering cone. In [30,31] it is said in the title "with variable ordering structure", but this has to do with the fact that the ordering cone is perturbed, not with the commonly used sense which will be the one considered in the present paper. Therefore, it is natural to understand if we can obtain the stability of the sets of weak solutions of VOP w.r.t. a VOS with perturbation of the feasible set.

The rest of the paper is organized as follows. In Section 2, we collect some notations, definitions and properties which will be used in the following sections. In Section 3, the continuity of a scalarizing functional based on the oriented distance, and defined by means of a general set-valued map is studied. In this section, we provide sufficient conditions for the continuity of this functional in terms of the commonly used hypotheses of upper and lower continuity in the sense of Hausdorff of the set-valued map. Then, in Section 4, we consider the case when the aforementioned functional is defined w.r.t. a VOS. In this case, we see that the usual notions of Hausdorff upper and lower continuity are very restrictive. Then, to deal with this problem, a suitable notion of continuity for cone-valued maps is proposed, which allows us to establish the continuity of the functional in this case. Finally, in Section 5, we give an application of our results to study convergence Painlevé-Kuratowski of the set of weak solutions. Illustrative examples are given throughout the paper.

2. Preliminaries

Let X and Y be real normed vector spaces. Given a set $A \subset Y$, we denote the interior, the closure, the boundary, the complement and the convex hull of A by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, A^c and $\operatorname{co} A$, respectively. The cone generated by A is denoted $\operatorname{cone} A$. It is said that A is solid if $\operatorname{int} A \neq \emptyset$, pointed if $A \cap (-A) \subset \{0\}$, and proper if $\{0\} \neq A \neq Y$.

Let $\mathcal{P}_0(Y)$ be the set of all nonempty subsets of Y. For every $A, B \in \mathcal{P}_0(Y)$ and $\lambda \in \mathbb{R}$, we denote

$$A + B = \{y_1 + y_2 : y_1 \in A, y_2 \in B\}, \quad \lambda A = \{\lambda y : y \in A\}.$$

A nonempty set $K \subset Y$ is a cone if $\lambda K \subset K$ for all $\lambda \geq 0$ ($0 \in K$). A cone K is convex if $K + K \subset K$.

Also, we denote by $B(y, r)$ the open ball of radius $r > 0$ centered at $y \in Y$. Recall that the distance of $y \in Y$ to a set A is given by $d(y, A) = \inf_{x \in A} \|x - y\|$, being $+\infty$ if $A = \emptyset$.

The following two lemmas will be used along the paper. The first one can be found in [32]. The proof of the second one is straightforward.

Lemma 2.1. *Let $A \in \mathcal{P}_0(Y)$, $y \in Y$ and $r \geq 0$. Then, $y \in \operatorname{cl}(A + B(0, r))$ if and only if $d(y, A) \leq r$.*

Lemma 2.2. *Let $K \subset Y$ and $y \in Y$. If K is a cone, then $d(ty, K) = td(y, K)$ for all $t > 0$.*

Next, the oriented distance function of Hiriart-Urruty [22] is introduced.

Definition 2.3. Let $A \subset Y$. The oriented distance $D(\cdot, A) : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined as follows:

$$D(y, A) = d(y, A) - d(y, A^c) = \begin{cases} d(y, A) & \text{if } y \in A^c \\ -d(y, A^c) & \text{if } y \in A. \end{cases}$$

It is considered that $D(y, \emptyset) = +\infty$ and $D(y, Y) = -\infty$.

In the sequel, we collect some basic properties of this function (see, for instance, [22,23,32]).

Lemma 2.4. *Let $A, B \in \mathcal{P}_0(Y)$, $A \neq Y$ and $y \in Y$. Then*

(i) $D(x, A) \in \mathbb{R}$ *for all* $x \in Y$, *and* $D(\cdot, A)$ *is Lipschitz of rank* 1.
(ii) $D(y, A) < 0 \Leftrightarrow y \in \operatorname{int} A$, $D(y, A) = 0 \Leftrightarrow y \in \operatorname{bd} A$, *and* $D(y, A) > 0 \Leftrightarrow y \notin \operatorname{cl} A$.
(iii) *If* $A \subset B$, *then* $D(y, B) \leq D(y, A)$.
(iv) $D(y, A) = D(-y, -A)$, *and* $D(y, A) = D(x + y, x + A)$, *for all* $x \in Y$.
(v) $D(y, A^c) = -D(y, A)$.
(vi) $D(y, A) \leq d(y, A)$, *and* $D(y, A) = d(y, A)$ *if* $\operatorname{int} A = \emptyset$ *or* $D(y, A) \geq 0$.

Given a convex cone K, we can define in Y an order relation $\leq_K$ as usual:

$$\forall x, y \in Y,\ x \leq_K y \Leftrightarrow y - x \in K.$$

From now on, the set-valued map $\mathcal{K} : Y \rightrightarrows Y$ denotes a variable ordering structure, named ordering map, which means that $\mathcal{K}(y)$ is a proper (i.e., $\{0\} \neq \mathcal{K}(y) \neq Y$) cone of Y, for all $y \in Y$. This map is used to compare elements on Y.

Indeed, given $a, b \in Y$, the following binary relations on Y are defined (see [7]):

$$a \leq_{\mathcal{K},1} b \Leftrightarrow b \in a + \mathcal{K}(a) \quad \text{and} \quad a \ll_{\mathcal{K},1} b \Leftrightarrow b \in a + \operatorname{int} \mathcal{K}(a).$$

For the sake of simplicity, they are denoted, respectively, $\leq_1$ and $\ll_1$. If $\mathcal{K}(y)$ is closed for all $y \in Y$, then it is clear by Lemma 2.4(ii) that $a \leq_1 b$ if and only if $D(b, a + \mathcal{K}(a)) \leq 0$. If $\mathcal{K}(y) = K$, for all $y \in Y$, then $\leq_1$ becomes $\leq_K$.

From now on, if N denotes a property of sets on Y, it is said that $\mathcal{K}$ is N-valued on Y is $\mathcal{K}(y)$ has the property N for all $y \in Y$.

In the literature, most of the authors consider that $\mathcal{K}$ is a convex cone-valued map, while some authors have recently considered that $\mathcal{K}$ is a general set-valued map (see [15,24]).

Next, we recall the concept of weak minimal element of a set (see, for instance, [1,7,10,33]), for which we will use the following natural condition.

Assumption A1. $\mathcal{K}(y)$ is a proper solid convex cone for all $y \in Y$.

Definition 2.5. Let $A \in \mathcal{P}_0(Y)$ and let Assumption A1 be satisfied. It is said that a point $a_0 \in A$ is a weak $\leq_1$-minimal element of A, denoted $a_0 \in \operatorname{WMin}(A, \leq_1)$, if $a \not\ll_1 a_0$ for all $a \in A \setminus \{a_0\}$, that is, if $a_0 \notin a + \operatorname{int} \mathcal{K}(a)$ for all $a \in A \setminus \{a_0\}$.

Given $f : X \to Y$ and $\emptyset \neq S \subset X$, recall that a point $x_0 \in S$ is said to be a weak minimal point of f on S, denoted by $x_0 \in W(f, S, \leq_1)$, if $f(x_0) \in \operatorname{WMin}(f(S), \leq_1)$. The following lemma was proved in [1, Theorem 5.5(b)].

Lemma 2.6. *Let $A \in \mathcal{P}_0(Y)$, $a_0 \in A$ and assume that Assumption A1 holds. Then the following statements are equivalent:*

(i) $a_0 \in \operatorname{WMin}(A, \leq_1)$.
(ii) $a_0 \notin a + \operatorname{int} \mathcal{K}(a)$, *for all* $a \in A$.
(iii) $D(a_0, a + \mathcal{K}(a)) \geq 0$, *for all* $a \in A$.

In consequence, $x_0 \in W(f, S, \leq_1)$ *if and only if* $D(f(x_0), f(x) + \mathcal{K}(f(x))) \geq 0$ *for all* $x \in S$.

As we have seen, the function $g_b^{\mathcal{K}} : Y \to \mathbb{R}$, $b \in Y$, defined as

$$g_b^{\mathcal{K}}(y) = D(b, y + \mathcal{K}(y)),\ \forall y \in Y, \tag{1}$$

appears in Lemma 2.6(iii). This function is also important to study, for instance, the closedness of sections $S(\leq_1, y_0) := \{y \in Y : y \leq_1 y_0\} = \{y \in Y : D(y_0, y + \mathcal{K}(y)) \leq 0\}$, the closedness of set $\operatorname{WMin}(A, \leq_1)$, Painlevé-Kuratowski convergence of the set of weak solutions (see Section 5), or stability of parametric problems [34]. These topics have been approached for a constant ordering map, but not for a general ordering map. For this reason, we think that the study of the continuity of $g_b^{\mathcal{K}}$ is an interesting and necessary question.

The following example shows that this function is not continuous in general.

Example 2.7. Let $Y = \mathbb{R}^2$, $b = (-1, 0)$, $y_0 = (0, 0)$, and let $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by

$$\mathcal{K}(y) = \begin{cases} \mathbb{R}^2_+ & \text{if } y_1 > 0 \\ K_1 & \text{if } y_1 \leq 0, \end{cases}$$

where $y = (y_1, y_2)$, and $K_1 = \{y \in \mathbb{R}^2 : y_2 \geq 0,\ y_1 + y_2 \geq 0\}$. Let us check that $g_b^{\mathcal{K}}$ is not continuous at y_0. Indeed, on the one hand, $g_b^{\mathcal{K}}(y_0) = d(b, K_1) = \sqrt{2}/2$. On the other hand, if $y = (y_1, y_2)$ with $y_1 > 0$ and $y_2 \geq 0$, then $g_b^{\mathcal{K}}(y) = D(b, y + \mathcal{K}(y)) = D(b, y + \mathbb{R}^2_+) = \sqrt{(y_1 + 1)^2 + y_2^2}$, which converges to 1 if $y \to (0, 0)$. Therefore, $g_b^{\mathcal{K}}$ is not continuous at y_0.

In the above example, the ordering map $\mathcal{K}$ is not continuous in any of the senses (i), (iii) or (iv) considered in Definition 2.8. Then, a natural question arises: Is $g_b^{\mathcal{K}}$ continuous under some kind of continuity of $\mathcal{K}$? The answer to this question is the main goal of this paper.

For this aim, we need first to focus on the analysis of the continuity of the function $g_b : X \to \mathbb{R}$, defined as

$$g_b(x) = D(b, F(x)), \quad \forall x \in X, \tag{2}$$

where $b \in Y$, and $F : X \rightrightarrows Y$ is a general set-valued map with $\emptyset \neq F(x) \neq Y$ for all $x \in X$. This is the target of the next section.

The continuity of g_b will depend on continuity hypotheses on the set-valued map F. Next, we remind the classical definitions of continuity for F (see [26, Definitions 2.5.1 and 2.5.12]).

Definition 2.8. Let $x_0 \in X$. The set-valued map $F : X \rightrightarrows Y$ is said to be

(i) upper continuous (u.c.) at x_0 if for any open set $V \subset Y$ with $F(x_0) \subset V$, there exists $\delta > 0$ such that $F(x) \subset V$, for all $x \in B(x_0, \delta)$.
(ii) lower continuous (l.c.) at x_0 if for all $y \in F(x_0)$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x) \cap B(y, \varepsilon) \neq \emptyset$, for all $x \in B(x_0, \delta)$.
(iii) Hausdorff upper continuous (H-u.c.) at x_0 if for all $\varepsilon > 0$, there exists $\delta > 0$, such that $F(x) \subset F(x_0) + B(0, \varepsilon)$, for all $x \in B(x_0, \delta)$.
(iv) Hausdorff lower continuous (H-l.c.) at x_0 if for all $\varepsilon > 0$, there exists $\delta > 0$, such that $F(x_0) \subset F(x) + B(0, \varepsilon)$, for all $x \in B(x_0, \delta)$.

We say that F is continuous (resp., H-continuous) at x_0 if F is u.c. and l.c. (resp., H-u.c. and H-l.c.) at x_0. We say that F is u.c. (resp., l.c., H-u.c., H-l.c., continuous, H-continuous), if F is u.c. (resp., l.c., H-u.c., H-l.c., continuous, H-continuous) at each point in X.

The following lemmas collect some relations among the continuity notions presented above (see, for instance, [26, page 59]).

Lemma 2.9. *If F is u.c. at x_0, then F is H-u.c. at x_0. The converse holds if $F(x_0)$ is compact.*

On the other hand, if F is H-l.c. at x_0, then F is l.c. at x_0, being the converse true if $F(x_0)$ is compact. But in general, the lower continuity of F at x_0 does not imply the H-lower continuity of F at x_0. Example 4.4 is a clear counter-example.

Lemma 2.10. *If $F_1, F_2 : X \rightrightarrows Y$ are l.c. (resp. H-u.c., H-l.c.) at x_0, then $F_1 + F_2$ is l.c. (resp. H-u.c., H-l.c.) at x_0.*

This result is not true for u.c. set-valued maps.

3. Continuity of the function g_b

In this section, we are going to study the continuity of g_b (see (2)) at a point $x_0 \in X$, for $b \in Y$.

In the following lemma, we collect two previous results (see [26, Proposition 2.5.14(iii)-(iv)]).

Lemma 3.1. *(i) F is H-l.c. at x_0 if and only if for every sequence $(x_n) \subset X$ with $(x_n) \to x_0$, and every sequence $(y_n) \subset F(x_0)$, there exists a sequence $(\bar{y}_n) \subset Y$ such that $(\bar{y}_n - y_n) \to 0$ and $\bar{y}_n \in F(x_n)$ for all n large enough.*

(ii) F is H-u.c. at x_0 if and only if for every sequence $((x_n, y_n)) \subset \operatorname{gr} F$ with $(x_n) \to x_0$, there exists a sequence $(\bar{y}_n) \subset F(x_0)$ such that $(y_n - \bar{y}_n) \to 0$.

In the following results, we will consider the next assumption.

Assumption A2. For every $x \in X$ one has $\emptyset \neq F(x) \neq Y$.

Theorem 3.2. *Let Assumption A2 be satisfied.*

(i) Suppose that $b \notin F(x_0)$. If F is H-l.c. at x_0, then g_b is upper semicontinuous (u.s.c.) at x_0.

(ii) Suppose that $b \notin \operatorname{cl} F(x_0)$. If F is H-u.c. at x_0, then g_b is lower semicontinuous (l.s.c.) at x_0.

(iii) Suppose that $b \notin \operatorname{cl} F(x_0)$. If F is H-continuous at x_0, then g_b is continuous at x_0.

Proof. (i) Let $\beta \in \mathbb{R}$, and $(x_n) \subset X$, such that $(x_n) \to x_0$ and $g_b(x_n) \geq \beta$ for all n. We have to prove that $g_b(x_0) \geq \beta$. By contradiction, assume that $g_b(x_0) < \beta$. As $b \notin F(x_0)$ we have

$$g_b(x_0) = D(b, F(x_0)) = d(b, F(x_0)) = \inf_{z \in F(x_0)} \|b - z\|.$$

Since $g_b(x_0) < \beta$, there exists $z_0 \in F(x_0)$ such that

$$g_b(x_0) \leq \|b - z_0\| < \beta. \tag{3}$$

Let $z_n = z_0$ for all n. As $(z_n) \subset F(x_0)$ and F is H-l.c. at x_0, by Lemma 3.1(i), there exists a sequence $(\bar{z}_n)$ with $\bar{z}_n \in F(x_n)$ and $(\bar{z}_n - z_n) \to 0$, i.e. $(\bar{z}_n) \to z_0$. Now, for every n, we have

$$\beta \leq g_b(x_n) = D(b, F(x_n)) \leq d(b, F(x_n)) = \inf_{z \in F(x_n)} \|b - z\| \leq \|b - \bar{z}_n\|,$$

and taking the limit, we derive that $\beta \leq \lim_{n\to\infty} \|b - \bar{z}_n\| = \|b - z_0\|$, which contradicts (3).

(ii) Let $\beta \in \mathbb{R}$, and $(x_n) \subset X$, such that $(x_n) \to x_0$ and $g_b(x_n) \leq \beta$ for all n. We have to prove that $g_b(x_0) \leq \beta$. By contradiction, assume that $g_b(x_0) > \beta$, and let $\alpha \in \mathbb{R}$ such that

$$\beta < \alpha < g_b(x_0). \tag{4}$$

Since $b \notin \operatorname{cl} F(x_0)$, there exists $\varepsilon > 0$ such that $B(b, \varepsilon) \cap F(x_0) = \emptyset$, and therefore

$$b \notin F(x_0) + B(0, \varepsilon). \tag{5}$$

As F is H-u.c. at x_0, from the definition it follows that there is $\delta > 0$ satisfying $F(x) \subset F(x_0) + B(0, \varepsilon)$ for all $x \in B(x_0, \delta)$, and taking into account that $(x_n) \to x_0$, there exists n_0 such that $x_n \in B(x_0, \delta)$ for all $n \geq n_0$, and so $F(x_n) \subset F(x_0) + B(0, \varepsilon)$, for all $n \geq n_0$. In view of (5) we derive that $b \notin F(x_n)$, for all $n \geq n_0$, and therefore

$$g_b(x_n) = D(b, F(x_n)) = d(b, F(x_n)) = \inf_{z \in F(x_n)} \|b - z\|, \quad \forall n \geq n_0.$$

Now, since $g_b(x_n) \leq \beta < \alpha$ for all n, there exists $z_n \in F(x_n)$ such that

$$\|b - z_n\| < \alpha, \quad \forall n \geq n_0. \tag{6}$$

As F is H-u.c. at x_0, by Lemma 3.1(ii) there exists a sequence $(\bar{z}_n) \subset F(x_0)$ such that

$$(z_n - \bar{z}_n) \to 0. \tag{7}$$

On the other hand, since $\bar{z}_n \in F(x_0)$ and $b \notin F(x_0)$, we have

$$\|b - \bar{z}_n\| \geq d(b, F(x_0)) = D(b, F(x_0)) = g_b(x_0), \quad \forall n \tag{8}$$

and taking into account (6) we have that

$$\|b - \bar{z}_n\| \leq \|b - z_n\| + \|z_n - \bar{z}_n\| < \alpha + \|z_n - \bar{z}_n\|, \quad \forall n \geq n_0. \tag{9}$$

By using (7) and (4), it follows that $\alpha + \|z_n - \bar{z}_n\| < g_b(x_0)$ for n large enough, and therefore in view of (9), $\|b - \bar{z}_n\| < g_b(x_0)$ for n large enough, in contradiction with (8).

(iii) It is an immediate consequence of parts (i) and (ii). $\square$

In what follows, we consider the set-valued map $F^c : X \rightrightarrows Y$ defined by $F^c(x) = F(x)^c$. This set-valued map F^c will be also very useful to study the continuity of g_b.

In the next results, we provide sufficient conditions for the upper and lower semicontinuity of g_b in terms of H-upper and H-lower continuity hypotheses on F^c, for $b \in F(x_0)$.

Theorem 3.3. *Let Assumption A2 be satisfied.*

(i) Suppose that $b \in F(x_0)$. If F^c is H-l.c. at x_0, then g_b is l.s.c. at x_0.
(ii) Suppose that $b \in \operatorname{int} F(x_0)$. If F^c is H-u.c. at x_0, then g_b is u.s.c. at x_0.
(iii) Suppose that $b \in \operatorname{int} F(x_0)$. If F^c is H-continuous at x_0, then g_b is continuous at x_0.

Proof. Define $G(x) := F^c(x)$ and $h_b(x) := D(b, G(x))$. Note that by Lemma 2.4(v)

$$g_b(x) = D(b, F(x)) = -D(b, F^c(x)) = -h_b(x), \quad \forall x \in X. \tag{10}$$

Then, we can apply Theorem 3.2(i)-(ii) to the pair (G, h_b) as follows.

(i) If $b \in F(x_0)$, then $b \notin G(x_0)$ and as $G = F^c$ is H-l.c. at x_0 by hypothesis, from Theorem 3.2(i) it follows that h_b is u.s.c. at x_0. Taking into account (10) we conclude that g_b is l.s.c. at x_0.

(ii) As $b \in \operatorname{int} F(x_0)$ and $[\operatorname{int}(F(x_0))]^c = \operatorname{cl}[F(x_0)^c]$, we deduce that $b \notin \operatorname{cl} G(x_0)$. Since $G = F^c$ is H-u.c at x_0, from Theorem 3.2(ii) we have that h_b is l.s.c. at x_0, and by (10) we derive that g_b is u.s.c. at x_0.

(iii) It is a direct consequence of (i) and (ii). □

In the next theorem we analyze the case when $b \in \operatorname{bd} F(x_0)$.

Theorem 3.4. *Let Assumption A2 be satisfied and suppose that $b \in \operatorname{bd} F(x_0)$.*

(i) If F is H-l.c. at x_0, then g_b is u.s.c. at x_0.
(ii) If F^c is H-l.c. at x_0, then g_b is l.s.c. at x_0.
(iii) If F and F^c are H-l.c. at x_0, then g_b is continuous at x_0.

Proof. (i) Let $\beta \in \mathbb{R}$, and $(x_n) \subset X$, such that $(x_n) \to x_0$ and $g_b(x_n) \geq \beta$ for all n. We have to prove that $g_b(x_0) \geq \beta$.

As $b \in \operatorname{bd} F(x_0)$, by Lemma 2.4(ii) one has $g_b(x_0) = D(b, F(x_0)) = 0$, so we have to prove that $0 \geq \beta$. By reasoning to the contrary assume that $\beta > 0$. Then

$$0 < \beta \leq g_b(x_n) = D(b, F(x_n)) = d(b, F(x_n)), \quad \forall n. \tag{11}$$

By hypothesis $b \in \operatorname{bd} F(x_0) = \operatorname{cl} F(x_0) \setminus \operatorname{int} F(x_0)$, and so there exists a sequence $(\bar{b}_n) \subset F(x_0)$ such that $(\bar{b}_n) \to b$. As F is H-l.c. at x_0, by Lemma 3.1(i) there exists a sequence $(z_n) \subset Y$ such that $(z_n - \bar{b}_n) \to 0$ and $z_n \in F(x_n)$. Taking into account (11) we have

$$0 < \beta \leq g_b(x_n) = d(b, F(x_n)) \leq \|b - z_n\| \leq \|b - \bar{b}_n\| + \|\bar{b}_n - z_n\|, \quad \forall n.$$

From here, since $\|b - \bar{b}_n\| \to 0$ and $\|\bar{b}_n - z_n\| \to 0$, we conclude that $0 < \beta \leq \lim_{n\to\infty} g_b(x_n) = 0$, which is a contradiction.

(ii) We apply part (i) to $G = F^c$ instead of F. This is possible because $b \in \operatorname{bd} G(x_0) = \operatorname{bd} F(x_0)$. Then we obtain that h_b is u.s.c. at x_0, where $h_b(x) = D(b, G(x))$. Now, as $g_b = -h_b$ by equation (10), we conclude that g_b is l.s.c. at x_0.

(iii) It is an immediate consequence of parts (i) and (ii). □

By using Theorem 3.2(iii), Theorem 3.3(iii) and Theorem 3.4(iii) we obtain the following result.

Theorem 3.5. *Let Assumption A2 be satisfied. If F and F^c are H-continuous at x_0, then g_b is continuous at x_0 for each $b \in Y$.*

To illustrate the above results we give an example.

Example 3.6. Let $Y = \mathbb{R}^2$ with the infinite norm $\|\cdot\|_\infty$, i.e., $\|(y_1, y_2)\|_\infty = \max\{|y_1|, |y_2|\}$. We denote $x_+ = \max\{0, x\}$ and $\bar{B}(0, r) = \{y \in \mathbb{R}^2 : \|y\|_\infty \leq r\}$, for $x \in \mathbb{R}$ and $r \geq 0$. The distances d and D in $\mathbb{R}^2$ are the ones associated to $\|\cdot\|_\infty$. Consider the maps $F_i : \mathbb{R} \rightrightarrows \mathbb{R}^2$, $i = 1, 2$, given by

$$F_1(x) = \begin{cases} \bar{B}(0, x_+) & \text{if } x \leq 2 \\ \bar{B}(0, x-1) & \text{if } x > 2, \end{cases} \qquad F_2(x) = \begin{cases} \bar{B}(0, x_+) & \text{if } x < 2 \\ \bar{B}(0, x-1) & \text{if } x \geq 2. \end{cases}$$

Let us observe that F_1, F_1^c, F_2 and F_2^c are H-continuous on $\mathbb{R} \setminus \{2\}$, F_1^c and F_2 are not H-u.c. at 2, and F_1 and F_2^c are not H-l.c. at 2. In Table 1, we present an scheme of several considered cases.

	Data: F, x_0, b	Facts	Apply Theor.	Concl. g_b is
a	F_2, 2, $(3,1)$	$b \notin F_2(x_0)$, F_2 is H-l.c.	3.2(i)	u.s.c.
b	F_1, 2, $(3,1)$	$b \notin \operatorname{cl} F_1(x_0)$, F_1 is H-u.c.	3.2(ii)	l.s.c.
c	F_1, 2, $(2,1)$	$b \in F_1(x_0)$, F_1^c is H-l.c.	3.3(i)	l.s.c.
d	F_2, 2, $(\frac{1}{2},0)$	$b \in \operatorname{int} F_2(x_0)$, F_2^c is H-u.c.	3.3(ii)	u.s.c.
e	F_2, 2, $(1,0)$	$b \in \operatorname{bd} F_2(x_0)$; F_2 is H-l.c.	3.4(i)	u.s.c.
f	F_1, 3, $(2,1)$	$b \in \operatorname{bd} F_1(x_0)$; F_1, F_1^c are H-l.c.	3.4(iii)	cont.

Table 1. Example 3.6.

Next, making some direct calculations, we obtain the expressions of the corresponding functions g_b, which corroborate the results given in Table 1:

(a) $F = F_2$, $x_0 = 2$ and $b = (3,1) \notin F_2(x_0)$:

$$g_b(x) = \begin{cases} \min\{3, 3-x\} & \text{if } x < 2 \\ 4 - x & \text{if } x \geq 2. \end{cases}$$

(b) $F = F_1$, $x_0 = 2$ and $b = (3,1) \notin \operatorname{cl} F_1(x_0)$:

$$g_b(x) = \begin{cases} \min\{3, 3-x\} & \text{if } x \leq 2 \\ 4 - x & \text{if } x > 2. \end{cases}$$

(c) $F = F_1$, $x_0 = 2$ and $b = (2,1) \in F_1(x_0)$:

$$g_b(x) = \begin{cases} \min\{2, 2-x\} & \text{if } x \leq 2 \\ 3 - x & \text{if } x > 2. \end{cases}$$

(d) $F = F_2$, $x_0 = 2$ and $b = (1/2, 0) \in \operatorname{int} F_2(x_0)$:

$$g_b(x) = \begin{cases} \min\{1/2, 1/2-x\} & \text{if } x < 2 \\ 3/2 - x & \text{if } x \geq 2. \end{cases}$$

(e) $F = F_2$, $x_0 = 2$ and $b = (1,0) \in \operatorname{bd} F_2(x_0)$:

$$g_b(x) = \begin{cases} \min\{1, 1-x\} & \text{if } x < 2 \\ 2 - x & \text{if } x \geq 2. \end{cases}$$

(f) $F = F_1$, $x_0 = 3$ and $b = (2,1) \in \operatorname{bd} F_1(x_0)$, one has that F_1 and F_1^c are H-l.c. at x_0. Function g_b is the same as in part (c).

In the next proposition, we relate the continuity properties of F and F^c. For this aim, we need a previous lemma.

Lemma 3.7. *([35, Lemma 1]) Let A, B, C subsets of Y and suppose that A is closed and convex and B is bounded. If $C + B \subset A + B$, then $C \subset A$.*

Proposition 3.8. *(i) If F is H-l.c. at x_0 and closed-valued and convex-valued on a neighbourhood of x_0, then F^c is H-u.c. at x_0.*

(ii) If F is H-u.c. at x_0 and $F(x_0)$ is closed and convex, then F^c is H-l.c. at x_0.

(iii) If F is H-continuous at x_0, closed-valued and convex-valued, then F^c is H-continuous at x_0.

Proof. (i) Given $\varepsilon > 0$, as F is H-l.c. at x_0, there exists $\delta > 0$ such that

$$F(x_0) \subset F(x) + B(0, \varepsilon), \quad \forall x \in B(x_0, \delta). \tag{12}$$

We can assume that F is closed-valued and convex-valued on $B(x_0, \delta)$.We affirm that

$$F^c(x) \subset F^c(x_0) + B(0, \varepsilon), \quad \forall x \in B(x_0, \delta),$$

which proves that F^c is H-u.c. at x_0. By reasoning to the contrary, assume that there exists $x_1 \in B(x_0, \delta)$ satisfying $F^c(x_1) \not\subset F^c(x_0)+B(0, \varepsilon)$. Then there exists $y_1 \in F^c(x_1)$ such that $y_1 \notin F^c(x_0)+B(0, \varepsilon)$. From here, we have $y_1 - u \notin F^c(x_0)$ for all $u \in B(0, \varepsilon)$, or equivalently, $y_1 - u \in F(x_0)$ for all $u \in B(0, \varepsilon)$. In consequence, $y_1 - B(0, \varepsilon) \subset F(x_0)$. As $-B(0, \varepsilon) = B(0, \varepsilon)$, we derive that $y_1 + B(0, \varepsilon) \subset F(x_0)$. By using (12) with $x = x_1$, it results that $F(x_0) \subset F(x_1) + B(0, \varepsilon)$, and therefore, $y_1 + B(0, \varepsilon) \subset F(x_1) + B(0, \varepsilon)$. As $F(x_1)$ is closed and convex, and $B(0, \varepsilon)$ is bounded, from Lemma 3.7 it follows that $y_1 \in F(x_1)$, which is a contradiction.

The proof of part (ii) is similar, so we omit it, and part (iii) is a direct consequence of parts (i) and (ii). □

Taking into account Proposition 3.8, and Theorems 3.3, 3.4 and 3.5, we obtain the following corollary.

Corollary 3.9. *Let Assumption A2 be satisfied.*

(i) If F is H-u.c. at x_0 and $F(x_0)$ is closed and convex, then g_b is l.s.c. at x_0 for each $b \in Y$.

(ii) If F is H-l.c. at x_0, closed-valued and convex-valued, then g_b is u.s.c. at x_0 for each $b \in Y$.

(iii) If F is H-continuous at x_0, closed-valued and convex-valued, then g_b is continuous at x_0 for each $b \in Y$.

Proof. (i) First assume that $b \notin F(x_0)$. As $F(x_0)$ is closed, one has $b \notin \operatorname{cl} F(x_0)$, and as F is H-u.c., by applying Theorem 3.2(ii) we get that g_b is l.s.c. at x_0.

Second, assume that $b \in F(x_0)$. As $F(x_0)$ is closed and convex and F is H-u.c. at x_0, by Proposition 3.8(ii) we derive that F^c is H-l.c. at x_0. Now, from Theorem 3.3(i) it follows that g_b is l.s.c. at x_0.

(ii) It is proved using the same ideas, but now we have to add a case. First, if $b \notin \operatorname{cl} F(x_0)$, by Theorem 3.2(i) we get that g_b is u.s.c. at x_0. Second, if $b \in \operatorname{bd} F(x_0)$, by Theorem 3.4(i) we obtain that g_b is u.s.c. at x_0. Third, if $b \in \operatorname{int} F(x_0)$, by Proposition 3.8(i) we derive that F^c is H-u.c. at x_0, and finally by Theorem 3.3(ii) we conclude that g_b is u.s.c. at x_0.

Part (iii) is a consequence of parts (i) and (ii). □

In the diagram presented in Figure 3 we collect all the relations between all the types of continuity of F, F^c and g_b studied in this section.

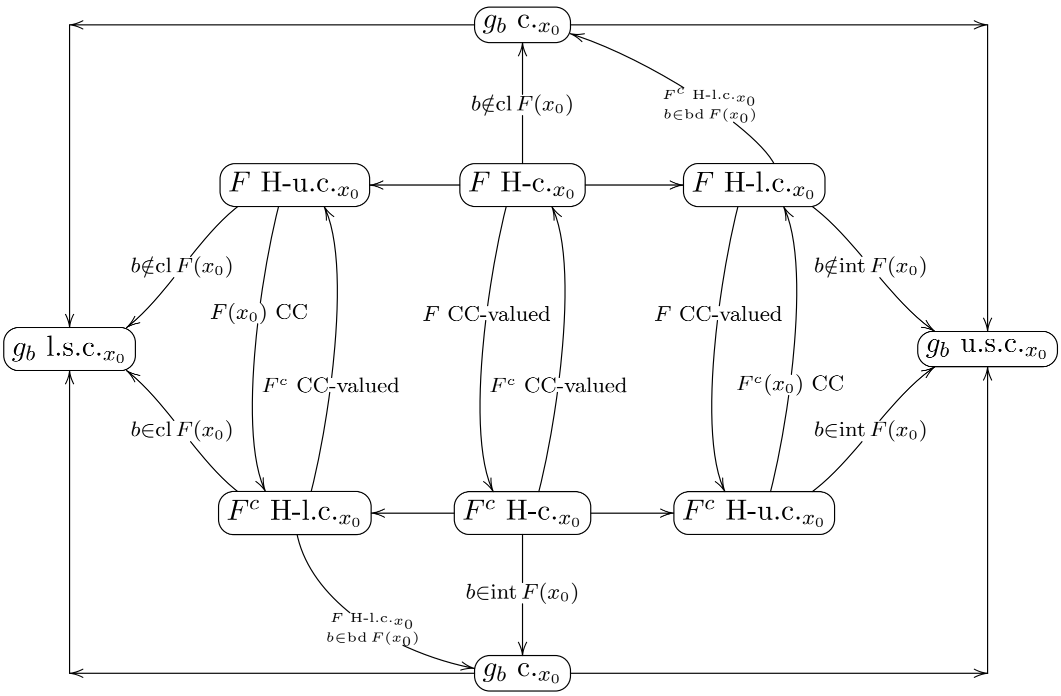

Figure 1. Connections between the types of continuity of F, F^c and g_b, under Assumption A2. Notation: H-l.c.$_{x_0}$, H-u.c.$_{x_0}$, l.s.c.$_{x_0}$, u.s.c.$_{x_0}$, H-c.$_{x_0}$ and c.$_{x_0}$, denote, respectively, H-l.c., H-u.c., l.s.c., u.s.c., H-continuity and continuity at x_0; CC refers to closed and convex.

Remark 1. (i) In [36, Corollary 5.1] it is proved that g_b is l.s.c. on X if F is u.c. on X, and nonempty and compact-valued. In contrast to this result, in Corollary 3.9(i) we use a weaker continuity assumption and we suppose that $F(x_0)$ is closed and convex instead of the compactness condition.

(ii) In [36, Corollary 5.2] the author establishes that g_b is u.s.c. on X if F is continuous on X, with nonempty, convex and compact values. This result is a direct consequence of Corollary 3.9(iii), because since F is compact-valued, we have that F is H-continuous, closed-valued and convex-valued, and by Corollary 3.9(iii), we get that g_b is continuous on X, a stronger result than the thesis of [36, Corollary 5.2]. Thus Corollary 3.9(iii) (also part (ii)) improves [36, Corollary 5.2].

Theorem 3.5 also improves [36, Corollary 5.2] (see Proposition 3.8(i)) and it is an alternative to [36, Corollary 5.1]. For instance, the set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ defined as $F(x) = \text{bd}\, B(0, x_+) := \{y \in \mathbb{R}^2 : \|y\|_\infty = x_+\}$ with $y = (y_1, y_2) \in Y$, satisfies the hypotheses of Theorem 3.5, but not the assumptions of [36, Corollary 5.2]. Besides, Theorem 3.5 can be also applied for G=F^c, but neither [36, Corollary 5.2] nor [36, Corollary 5.1] can be applied in this case. Note that G is neither compact nor convex-valued. Note also that Corollary 3.9(iii) cannot be applied to G.

4. Continuity of the function $g_b^{\mathcal{K}}$

In this section, we are going to study the continuity of the function $g_b^{\mathcal{K}}$ at a point $y_0 \in Y$ (see (1)), for $b \in Y$ and by assuming that $\mathcal{K} : Y \rightrightarrows Y$ is an ordering map ($\mathcal{K}(y)$ is a proper cone, for all $y \in Y$), in terms of some type of continuity of $\mathcal{K}$.

As we will see in the next propositions, the continuity notions given in Definition 2.8 are not appropriate for cone-valued maps, as it was pointed out by Eichfelder [1, pag. 67], because the continuity at a point entails strong restrictions to the values of $\mathcal{K}$ on a neighbourhood of the point.

Proposition 4.1. *It follows that $\mathcal{K}$ is H-u.c. at y_0 if and only if there exists $\delta > 0$ such that $\mathcal{K}(y) \subset \operatorname{cl} \mathcal{K}(y_0)$ for all $y \in B(y_0, \delta)$.*

Proof. ($\Leftarrow$) It is obvious.

($\Rightarrow$) By contradiction. Assume that for all n, there exists $y_n \in B(y_0, 1/n)$ such that $\mathcal{K}(y_n) \not\subset \operatorname{cl} \mathcal{K}(y_0)$. So, there exists $z_n \in \mathcal{K}(y_n)$ such that $z_n \notin \operatorname{cl} \mathcal{K}(y_0)$. Let ε_0 be a fixed positive number. As $\mathcal{K}$ is H-u.c. at y_0, there exists $\delta > 0$ such that

$$\mathcal{K}(y) \subset \mathcal{K}(y_0) + B(0, \varepsilon_0), \quad \forall y \in B(y_0, \delta). \tag{13}$$

For this $\delta > 0$, there exists n such that $y_n \in B(y_0, \delta)$ and $z_n \in \mathcal{K}(y_n) \setminus \operatorname{cl} \mathcal{K}(y_0)$. Let $\eta := d(z_n, \operatorname{cl} \mathcal{K}(y_0)) = d(z_n, \mathcal{K}(y_0))$, and $t_0 = \varepsilon_0/\eta$. Then, by Lemma 2.2

$$d(tz_n, \mathcal{K}(y_0)) = td(z_n, \mathcal{K}(y_0)) = t\eta > \varepsilon_0, \quad \forall t > t_0.$$

This implies that $tz_n \notin \mathcal{K}(y_0) + B(0, \varepsilon_0)$ by Lemma 2.1, and as $\mathcal{K}(y_n)$ is a cone, we have $tz_n \in \mathcal{K}(y_n)$, which contradicts (13). □

This proposition extends [1, Lemma 3.28], which requires Y to be a reflexive Banach space and $\mathcal{K}(y_0)$ a convex cone.

Proposition 4.2. *It follows that $\mathcal{K}$ is H-l.c. at y_0 if and only if there exists $\delta > 0$ such that $\mathcal{K}(y_0) \subset \operatorname{cl} \mathcal{K}(y)$ for all $y \in B(y_0, \delta)$.*

Proof. ($\Leftarrow$) It is obvious.

($\Rightarrow$) It is similar to the proof of the previous proposition. By contradiction, if the conclusion is false, then for all n there exist $y_n \in B(y_0, 1/n)$ and $z_n \in \mathcal{K}(y_0) \setminus \operatorname{cl} \mathcal{K}(y_n)$. Given a fixed $\varepsilon_0 > 0$, as $\mathcal{K}$ is H-l.c. at y_0, there exists $\delta > 0$ such that

$$\mathcal{K}(y_0) \subset \mathcal{K}(y) + B(0, \varepsilon_0), \quad \forall y \in B(y_0, \delta). \tag{14}$$

Then there exists n such that $y_n \in B(y_0, \delta)$ and $z_n \in \mathcal{K}(y_0) \setminus \operatorname{cl} \mathcal{K}(y_n)$. Let $\eta := d(z_n, \operatorname{cl} \mathcal{K}(y_n)) = d(z_n, \mathcal{K}(y_n))$, and $t_0 = \varepsilon_0/\eta$. Then, by Lemma 2.2

$$d(tz_n, \mathcal{K}(y_n)) = t\eta > \varepsilon_0, \quad \forall t > t_0,$$

and so $tz_n \notin \mathcal{K}(y_n) + B(0, \varepsilon_0)$ by Lemma 2.1, and $tz_n \in \mathcal{K}(y_0)$ since $\mathcal{K}(y_0)$ is a cone. This fact contradicts (14). □

To overcome the drawback explained before, for the ordering map $\mathcal{K}$, we consider the following continuity notions (see [27], where the notions are given for an upper and lower continuous set-valued map F).

We denote the closed unit ball in Y by $\mathbb{B}$. Given $M \in \mathcal{P}_0(Y)$, we denote by $\mathcal{K}_M$ the set-valued map $\mathcal{K}_M : Y \rightrightarrows Y$ defined by $\mathcal{K}_M(y) = \mathcal{K}(y) \cap M$. Given $\varepsilon > 0$ and a cone $A \in \mathcal{P}_0(Y)$, we denote $C_\varepsilon(A) := \{y \in Y : d(y, A) < \varepsilon\|y\|\} \cup \{0\}$ (conic ε-neighbourhood of A).

Definition 4.3. (i) It is said that $\mathcal{K}$ is cosmically H-u.c. (resp., H-l.c., H-continuous, u.c., l.c.) at y_0 if $\mathcal{K}_{\mathbb{B}} : Y \rightrightarrows Y$ is H-u.c. (resp., H-l.c., H-continuous, u.c., l.c.) at y_0.

(ii) It is said that $\mathcal{K}$ is conically u.c. at y_0, if for all $\varepsilon \in (0, 1)$, there exists $\delta > 0$ such that $\mathcal{K}(y) \subset C_\varepsilon(\mathcal{K}(y_0))$ for all $y \in B(0, \delta)$.

These notions seem to be more suitable to study continuity properties for a cone-valued map.

Example 4.4. Let $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by

$$\mathcal{K}(y_1, y_2) = \begin{cases} \mathbb{R}_+ \times \{0\} & \text{if } \ y_2 = 0 \\ \operatorname{cone}\operatorname{co}\{(1, |y_2|), (1, |y_2|/2)\} & \text{if } \ y_2 \neq 0, \end{cases}$$

and $y_0 = (0, 0)$. It follows that $\mathcal{K}$ is neither u.c. nor H-u.c. nor H-l.c. (even though it is l.c.) at y_0. However, $\mathcal{K}$ is cosmically continuous and cosmically H-continuous at y_0 as it can be easily checked.

Proposition 4.5. *The following assertions hold.*

(i) If $\mathcal{K}$ is cosmically u.c. at y_0, then $\mathcal{K}$ is cosmically H-u.c. at y_0.

(ii) If $\mathcal{K}$ is cosmically u.c. at y_0, then $\mathcal{K}$ is conically u.c. at y_0. Conversely, if $\mathcal{K}$ is conically u.c. at y_0 and $\mathcal{K}(y_0)$ is locally compact, then $\mathcal{K}$ is cosmically u.c. at this point.

Part (i) follows from the definition and Lemma 2.9, and part (ii) is proved in [27, Proposition 2.1(ii), (iv)].

In [27], it is proved that $\mathcal{K}$ is cosmically u.c. at y_0 if and only if $\mathcal{K}_M$ is cosmically u.c. at y_0 for every bounded closed set $M \in \mathcal{P}_0(Y)$. Next, we state a result of this kind.

Proposition 4.6. *The following statements are equivalent:*

(i) $\mathcal{K}$ is cosmically H-u.c. (resp., H-l.c.) at y_0.

(ii) $\mathcal{K}_{r\mathbb{B}}$ is H-u.c. (resp., H-l.c.) at y_0 for some $r > 0$.

(iii) $\mathcal{K}_{r\mathbb{B}}$ is H-u.c. (resp., H-l.c.) at y_0 for all $r > 0$.

The equivalences (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) are clear because $\mathcal{K}_{r\mathbb{B}} = r\,\mathcal{K}_{\mathbb{B}}$.

Remark 2. The results and definitions above can be extended straightforward for a cone-valued map $F : X \rightrightarrows Y$, but we have preferred to keep the same framework along the section.

Now, we can state the main result of this section. For this aim, we need a previous lemma.

Lemma 4.7. *If $\mathcal{K}$ and $\mathcal{K}^c$ are cosmically H-continuous at y_0, then we have that*

(i) $g^{\mathcal{K}}_{b,r}(y) := D(b, y + \mathcal{K}_{r\mathbb{B}}(y))$ and $g^{\mathcal{K}^c}_{b,r}(y) := D(b, y + \mathcal{K}^c_{r\mathbb{B}}(y))$ are continuous at y_0,

for all $r > 0$, and

(ii) there exist $r, \delta > 0$ such that

$$g_b^{\mathcal{K}}(y) = g_{b,r}^{\mathcal{K}}(y), \quad \forall y \in B(y_0, \delta).$$

(Here $\mathcal{K}_{r\mathbb{B}}^c(y) = \mathcal{K}(y)^c \cap (r\mathbb{B})$).

Proof. By Proposition 4.6, we know that $\mathcal{K}$ (respectively, $\mathcal{K}^c$) is cosmically H-continuous at y_0 if and only if $\mathcal{K}_{r\mathbb{B}}$ (respectively, $\mathcal{K}_{r\mathbb{B}}^c$) is H-continuous at y_0, for all $r > 0$. Then, by Theorem 3.5, we have that $g_{b,r}^{\mathcal{K}}$ and $g_{b,r}^{\mathcal{K}^c}$ are continuous at y_0, for all $r > 0$, since $H_r(y) := y + \mathcal{K}_{r\mathbb{B}}(y)$ and $H_r^c(y) := y + \mathcal{K}_{r\mathbb{B}}^c(y)$ are H-continuous at y_0 by Lemma 2.10, so part (i) is proved.

For the second part, observe that we always have that $g_b^{\mathcal{K}}(y) \leq g_{b,r}^{\mathcal{K}}(y)$, for all $y \in Y$ by Lemma 2.4(iii), since $y + \mathcal{K}_{r\mathbb{B}}(y) \subset y + \mathcal{K}(y)$, so we only have to prove the reciprocal inequality.

Thus, suppose by reasoning to the contrary that the inequality "$\geq$" does not hold for any $r, \delta > 0$. Then, for every $n \in \mathbb{N} \setminus \{0\}$, choosing $r = n$ and $\delta = \frac{1}{n}$, there exists $y_n \in B\left(y_0, \frac{1}{n}\right)$ such that

$$g_b^{\mathcal{K}}(y_n) < g_{b,n}^{\mathcal{K}}(y_n). \tag{15}$$

It follows that $y_n \to y_0$. Then, there exists $n_0 \in \mathbb{N}$ such that

$$b - y_n \in n_0\mathbb{B} \subset n \operatorname{int} \mathbb{B}, \quad \forall n > n_0. \tag{16}$$

We define the sets A_1 and A_2 as follows:

$$A_1 := \{n \in \mathbb{N} : b - y_n \notin \mathcal{K}(y_n) \cap (n\mathbb{B})\} \text{ and } A_2 := \mathbb{N} \setminus A_1.$$

At least one of the sets A_1 or A_2 is infinite, so we have two possibilities: A_1 is infinite or A_2 is infinite.

Case 1. A_1 is infinite. Without loss of generality we can suppose that $b - y_n \notin \mathcal{K}(y_n) \cap (n\mathbb{B})$, for all n. Thus, by (16), we have $b - y_n \notin \mathcal{K}(y_n)$, for all $n > n_0$.

Then, $g_b^{\mathcal{K}}(y_n) = d(b, y_n + \mathcal{K}(y_n))$ and $g_{b,n}^{\mathcal{K}}(y_n) = d(b, y_n + \mathcal{K}_{n\mathbb{B}}(y_n))$, for all $n > n_0$. So by (15) we have for all $n > n_0$ that

$$\inf_{a \in \mathcal{K}(y_n)} \|b - y_n - a\| < \inf_{a \in \mathcal{K}_{n\mathbb{B}}(y_n)} \|b - y_n - a\|.$$

Thus, for each $n > n_0$, there exists $\bar{a}_n \in \mathcal{K}(y_n)$ such that

$$\|b - y_n - \bar{a}_n\| < \inf_{a \in \mathcal{K}(y_n) \cap (n\mathbb{B})} \|b - y_n - a\|,$$

so in particular we deduce that $\bar{a}_n \notin n\mathbb{B}$, for all $n > n_0$. Hence, $\|\bar{a}_n\| \to +\infty$. Then, we have for all $n > n_0$ that

$$\begin{aligned} \|b - y_n - \bar{a}_n\| &< d(b, y_n + \mathcal{K}_{n\mathbb{B}}(y_n)) \\ &\leq d(b, y_n + \mathcal{K}_{\mathbb{B}}(y_n)) = D(b, y_n + \mathcal{K}_{\mathbb{B}}(y_n)) = g_{b,1}^{\mathcal{K}}(y_n) \end{aligned} \tag{17}$$

(the last two inequalities are true because $y_n + \mathcal{K}(y_n) \cap \mathbb{B} \subset y_n + \mathcal{K}_{n\mathbb{B}}(y_n)$ for all n and $b \notin y_n + \mathcal{K}(y_n) \cap \mathbb{B}$ since $b \notin y_n + \mathcal{K}_{n\mathbb{B}}(y_n)$). We know by part (i) that $g_{b,1}$ is continuous at y_0, so the right hand side of (17) tends to $g^{\mathcal{K}}_{b,1}(y_0) < +\infty$. Thus, we reach a contradiction since the expression on the left hand side tends to $+\infty$.

Case 2. A_2 is infinite. Without loss of generality we can suppose that $b - y_n \in \mathcal{K}(y_n) \cap (n\mathbb{B})$, for all n. Therefore, in this case we have $g^{\mathcal{K}}_b(y_n) = -d(b, y_n + \mathcal{K}^c(y_n))$ and $g^{\mathcal{K}}_{b,n}(y_n) = -d(b, y_n + (\mathcal{K}_{n\mathbb{B}}(y_n))^c)$, for all n. So, in view of (15), for each n we have that

$$\inf_{a \in \mathcal{K}^c(y_n)} \|b - y_n - a\| > \inf_{a \in (\mathcal{K}_{n\mathbb{B}}(y_n))^c} \|b - y_n - a\|.$$

Thus, for each n, there exists $\bar{a}_n \in (\mathcal{K}_{n\mathbb{B}}(y_n))^c = (\mathcal{K}(y_n) \cap (n\mathbb{B}))^c$ such that

$$\|b - y_n - \bar{a}_n\| < \inf_{a \in \mathcal{K}^c(y_n)} \|b - y_n - a\|. \tag{18}$$

Hence, we deduce that $\bar{a}_n \notin \mathcal{K}^c(y_n)$, i.e., $\bar{a}_n \in \mathcal{K}(y_n)$, and since $\bar{a}_n \notin \mathcal{K}(y_n) \cap (n\mathbb{B})$, we derive that $\bar{a}_n \notin n\mathbb{B}$, for all n, so $\|\bar{a}_n\| \to +\infty$. Therefore, using (18) and the facts $b - y_n \notin \mathcal{K}^c(y_n)$ and $y_n + \mathcal{K}^c(y_n) \cap \mathbb{B} \subset y_n + \mathcal{K}^c(y_n)$ one has

$$\begin{aligned}
\|b - y_n - \bar{a}_n\| &< d(b, y_n + \mathcal{K}^c(y_n)) = D(b, y_n + \mathcal{K}^c(y_n)) \\
&\leq D(b, y_n + \mathcal{K}^c(y_n) \cap \mathbb{B}) \\
&= D(b, y_n + \mathcal{K}^c_{\mathbb{B}}(y_n)) = g^{\mathcal{K}^c}_{b,1}(y_n), \quad \forall n,
\end{aligned}$$

and we reach a contradiction, since the left hand side of the statement above tends to $+\infty$ and the right hand side tends to $g^{\mathcal{K}^c}_{b,1}(y_0) = D(b, y_0 + \mathcal{K}^c_{\mathbb{B}}(y_0)) < +\infty$, because the function $g^{\mathcal{K}^c}_{b,1}$ is continuous at y_0 by part (i).

The proof is complete. □

Theorem 4.8. *If $\mathcal{K}$ and $\mathcal{K}^c$ are cosmically H-continuous at y_0, then $g^{\mathcal{K}}_b$ is continuous at y_0 for each $b \in Y$.*

Proof. The proof is almost straighforward by applying Lemma 4.7. Indeed, by Lemma 4.7(i), we know that $g^{\mathcal{K}}_{b,r}$ is continuous at y_0, for all $r > 0$, and by Lemma 4.7(ii) we know that there exist $r, \delta > 0$ such that $g^{\mathcal{K}}_b(y) = g^{\mathcal{K}}_{b,r}(y)$, for all $y \in B(y_0, \delta)$, so $g^{\mathcal{K}}_b$ is continuous at y_0, as we wanted to prove. □

We give an example to illustrate the main result.

Example 4.9. Let $Y = \mathbb{R}^2$ with the Euclidean norm, $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ defined by

$$\mathcal{K}(y) = \begin{cases} \operatorname{cone}\{(1,0)\} & \text{if } y_2 = 0 \\ \operatorname{cone}\operatorname{co}\{(1,|y_2|),(1,0)\} & \text{if } y_2 \neq 0, \end{cases}$$

where $y = (y_1, y_2)$ and $b = (b_1, b_2)$. It is easy to check that $\mathcal{K}$ and $\mathcal{K}^c$ are cosmically H-continuous on $\mathbb{R}^2$. In consequence, by Theorem 4.8 it follows that $g^{\mathcal{K}}_b$ is continuous on $\mathbb{R}^2$. After some calculations we obtain the following expression:

$$g_b^{\mathcal{K}}(y) = \begin{cases} \|y-b\| & \text{if } y_2 \geq b_2 \text{ and } y_1 \geq b_1 \\ y_2 - b_2 & \text{if } y_2 \geq b_2 \text{ and } y_1 < b_1 \\ \|y-b\| & \text{if } y_2 < b_2 \text{ and } y_1 \geq f_0 \\ \frac{|y_2-(y_1-b_1)|y_2|-b_2|}{\sqrt{y_2^2+1}} & \text{if } y_2 < b_2 \text{ and } f_1 < y_1 < f_0 \\ -\min\left\{\frac{|y_2-(y_1-b_1)|y_2|-b_2|}{\sqrt{y_2^2+1}}, b_2-y_2\right\} & \text{if } y_2 < b_2 \text{ and } y_1 \leq f_1, \end{cases}$$

where $f_0 := b_1 + |y_2|(b_2 - y_2)$ and $f_1 := \frac{y_2-b_2}{|y_2|}$. Note that if $y_2 = 0$, then $f_1 = -\infty$ and $g_b^{\mathcal{K}}(y) = b_2 - y_2$. Observe also that if $y_2 < b_2$ and $f_1 < y_1 < f_0$, then $g_b^{\mathcal{K}}(y) = d(b, r_y)$, where r_y is the line of slope $|y_2|$ through the point (y_1, y_2). The point (f_0, y_2) is the point where the perpendicular line to r_y through the point (b_1, b_2) cuts to the line $(0, y_2) + t(1, 0)$, $t \in \mathbb{R}$.

In [4, Theorem 1.57] it is studied the continuity of a Gerstewitz's scalarization for a VOS, but let us note that it is not applicable to our example because of two reasons: (i) $\operatorname{int}\mathcal{K}(y) = \emptyset$ for $y_2 = 0$ and (ii) $\mathcal{K}$ is not u.c. on $Y \times Y$ and $W(y) := Y \setminus \operatorname{int}\mathcal{K}(y)$ either. The same happens with [6, Theorem 2.1] since is the same result as [4, Theorem 1.57]. Theorem 2.1 in [5] is not applicable either since it requires that $\mathcal{K}$ is continuous. [36, Corollaries 5.1 and 5.2] cannot be applied to $F(y) = y + \mathcal{K}(y)$ because $F(y)$ is not compact.

To the best of our knowledge, Theorem 4.8 is the first result about the continuity of a scalarization w.r.t. a VOS under some general assumptions because the results given in [6, Theorem 2.1], [4, Theorem 1.57] and [5, Theorem 2.1] use, as we have seen in Propositions 4.1 and 4.2, restrictive notions of continuity for a cone-valued map.

Remark 3. Theorem 4.8 can be extended straightforward for a cone-valued map $F : X \rightrightarrows Y$, obtaining the following result: If F is cone-valued with $\emptyset \neq F(x) \neq Y$ for all $x \in X$ and F and F^c are cosmically H-continuous at x_0, then the function $g_b(x) = D(b, F(x))$ is continuous at x_0 for each $b \in Y$.

5. Convergence of the sets of weak minimal solutions

In this section we give an application to state the Painlevé-Kuratowski convergence of the sets of weak minimal solutions when the feasible set is perturbed.

Let (A_n) be a sequence of subsets of the normed space Y. Recall that (see [26])

$$\operatorname{Ls} A_n := \Big\{y \in Y : y = \lim_{k\to+\infty} y_{n_k},\ y_{n_k} \in A_{n_k},\ (n_k) \text{ a subsequence of } (n)\},$$
$$\operatorname{Li} A_n := \Big\{y \in Y : y = \lim_{n\to+\infty} y_n,\ y_n \in A_n,\ \text{for all large } n\Big\}.$$

The set $\operatorname{Ls} A_n$ is called the upper limit of the sequence (A_n), while the set $\operatorname{Li} A_n$ is called the lower limit of (A_n). We say that a sequence (A_n) converges in the sense of Painlevé-Kuratowski to the set A if $\operatorname{Ls} A_n \subset A \subset \operatorname{Li} A_n$, and we denote this convergence by $A_n \xrightarrow{K} A$. The inclusion $\operatorname{Ls} A_n \subset A$ is known as the upper part of convergence, while the inclusion $A \subset \operatorname{Li} A_n$ is the lower part of the convergence.

Lemma 5.1. *Assume that $\mathcal{K}$ and $\mathcal{K}^c$ are cosmically H-continuous. Then the function $\widetilde{D}_{\mathcal{K}} : Y \times Y \to \mathbb{R}$ given by $\widetilde{D}_{\mathcal{K}}(u, y) = D(u, y + \mathcal{K}(y))$ is continuous on $Y \times Y$.*

Proof. Let us prove that $\widetilde{D}_{\mathcal{K}}$ is continuous at $(u_0, y_0) \in Y \times Y$. We know that the function $y \mapsto D(b, y+\mathcal{K}(y))$ is continuous on Y for all $b \in Y$ by Theorem 4.8. So, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|D(u_0, y + \mathcal{K}(y)) - D(u_0, y_0 + \mathcal{K}(y_0))| < \varepsilon/2$ for all $y \in B(y_0, \delta)$. Then using that $D(\cdot, A)$ is Lipschitz of constant 1 for all set $A \in \mathcal{P}_0(Y)$ (see Lemma 2.4(i)), we have

$$\begin{aligned} |D(u, y + \mathcal{K}(y)) - D(u_0, y_0 + \mathcal{K}(y_0))| & \\ \leq |D(u, y + \mathcal{K}(y)) - D(u_0, y + \mathcal{K}(y))| & \\ + |D(u_0, y + \mathcal{K}(y)) - D(u_0, y_0 + \mathcal{K}(y_0))| & \\ \leq \|u - u_0\| + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon & \end{aligned}$$

for all $u \in B(u_0, \varepsilon/2)$ and $y \in B(y_0, \delta)$. □

Next, we establish the lower part of the Painlevé-Kuratowski convergence of the sets of weak minimal solutions for perturbed vector optimization problems.

Theorem 5.2. *Let S_n, S be nonempty subsets of X. Let Assumption A1 be satisfied. Assume the following conditions.*

(i) $\mathcal{K}$ *and* $\mathcal{K}^c$ *are cosmically H-continuous on* Y.
(ii) $f : X \to Y$ *is continuous.*
(iii) $S_n \xrightarrow{K} S$.

Then,

$$\operatorname{Ls} W(f, S_n, \leq_1) \subset W(f, S, \leq_1).$$

Proof. Let $x_0 \in \operatorname{Ls} W(f, S_n, \leq_1)$. Then there exists a subsequence (n_k) of (n) such that $x_{n_k} \in W(f, S_{n_k}, \leq_1)$ and $x_{n_k} \to x_0$. As by hypothesis $\operatorname{Ls} S_n \subset S$ and $x_{n_k} \in S_{n_k}$, it follows that $x_0 \in S$.

On the one hand, since $x_{n_k} \in W(f, S_{n_k}, \leq_1)$ by using Lemma 2.6 we derive

$$D(f(x_{n_k}), f(x) + \mathcal{K}(f(x))) \geq 0, \quad \forall x \in S_{n_k}. \tag{19}$$

On the other hand, given $z \in S$, as by hypothesis $S \subset \operatorname{Li} S_n$, there exists a sequence $(z_n) \subset X$ with $z_n \in S_n$ and $z_n \to z$. As $z_{n_k} \in S_{n_k}$, from (19) one has

$$D(f(x_{n_k}), f(z_{n_k}) + \mathcal{K}(f(z_{n_k}))) \geq 0, \quad \forall k \in \mathbb{N}. \tag{20}$$

As f is continuous on X, by applying Lemma 5.1 we deduce that the function $\psi : X \times X \to \mathbb{R}$ given by $\psi(x, v) = \widetilde{D}_{\mathcal{K}}(f(x), f(v)) = D(f(x), f(v)+\mathcal{K}(f(v)))$ is continuous since $\psi = \widetilde{D}_{\mathcal{K}} \circ (f, f)$. Therefore taking the limit when $k \to \infty$ in (20) it results that $\widetilde{D}_{\mathcal{K}}(f(x_0), f(z)) \geq 0$, and this is valid for all $z \in S$. By Lemma 2.6, we derive that $x_0 \in W(f, S, \leq_1)$. □

6. Conclusions

We have focused on the study of the continuity of a scalarizing function based on the oriented distance of Hiriart-Urruty w.r.t. a VOS, which is very useful to characterize minimal elements of a VOP. For this aim, firstly, we have provided a complete study of sufficient conditions for the continuity of this functional w.r.t. a general set-valued map, under continuity hypotheses on this set-valued map, in the sense of Hausdorff. In the particular case when we consider a VOS, we have shown that the requirement of the usual Hausdorff (lower or upper) continuity on the VOS is too strong. We have overcome this drawback, and we have obtained sufficient conditions for the continuity of the scalarizing function in study, under weaker continuity hypotheses on the VOS. Finally, under suitable conditions, Painlevé-Kuratowski convergence of the sets of weak minimal solutions for a perturbed VOP has been studied. In the future, it will be of interest to study continuity for other extensions of the oriented distance w.r.t. a VOS, the closedness of the set of weak minimal solutions of a set, stability for the sets of solutions of a set optimization problem, parametric optimization, etc.

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