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Market and model risks: a feasible joint estimate methodology

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Abstract
The increasing complexity of stochastic models used to describe the behavior of asset returns along with the practical difficulty of defining suitable hedging strategies are relevant factors that compromise the soundness and quality of risk measurement models. In this paper we define the risk model as the mispricing a consequence of using an inadequate model to describe asset behavior and we develop a least-squares Monte Carlo methodology to estimate market and model risk simultaneously. The results show that at different confidence levels and time horizons the proposed methodology to estimate the market and model risks has a greater joint explanatory power than the isolated estimate of market risk.

Keywords Model risk · Simulation model · Stochastic process · Monte Carlo · Least-squares

JEL Classification C15 · C35 · C51 · C52

Introduction

Basel (2009a) is a directive requiring financial institutions to quantify model risk taking into account two risks: the risk associated with a potentially incorrect valuation and the risk associated with using unobservable calibration parameters. Basel (2009b) are recommendations on controlling models and operational risk among

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which they indicate that model validation requires a sensitivity analysis performed to assess the impact of variations on model parameters. In practice, model parameter calibration is quite approximate and does not always yield meaningful data. The financial industry has come to no consensus on the risk model. Hénaff and Martini (2011) describes the state of the art of model risk and emphasizes the problem of identifying which model is the most appropriate. Hénaff and Martini (2011) argues that simulation studies provide the most comprehensive assessment of model risk and that we need a methodology to measure model risk considering wrong parameters, each model's probability, and a risk estimate as VaR (Value at Risk). Derman (1996) is the first systematic description of model risk:

- Inapplicability of model: a mathematical model may not be relevant to describe the problem at hand.
- Incorrect model: the possibility of using a model that does not accurately describe the situation being modeled. This is the most common interpretation of model risk.
- Correct model/incorrect solutions: these can lead to inconsistent pricing of related claims, among others.
- Correct model/inappropriate use: related to an inaccurate numerical solution of an otherwise correct model (e.g., the risk related to Monte Carlo calculations with too few simulations).
- Badly approximated solutions: this appears when numerical methods are used to solve a model.
- Software and hardware errors.
- Unstable data: financial data is of notoriously poor quality. Models must be robust with respect to errors in input data.

Cont (2006) distinguishes between uncertainty and risk. When a risk manager is not able to attribute a precise probability to future outcomes this is called uncertainty. By contrast when we can specify a unique probability measurement of future outcomes or ambiguity it is called risk. The econometric models specify a probability measure pursuant to the historical evolution of market prices, while pricing models use a risk neutral probability which relates various instruments' prices in an arbitrage-free manner. If the market is complete, then neutral probability is unique and defined by the historical evolution of prices; if the market is incomplete, then neutral probability is not unique. In these more realistic cases, we have to select a model compatible with market prices of underlying asset and hedging instruments. Often, however, the underlying values are not observable (e.g. the market value of the assets in the Merton (1974) model), are illiquid (OTC) or depend on a set of parameters (stochastic, alpha, beta and rho model, the SABR model).

Cont (2006) points out two ways to measure risk models: worst-case approaches and averaging model that incorporates model uncertainty into estimates using Monte Carlo algorithms.

Gupta et al. (2010) notes, like Cont (2006), that the calibration may use data for only the underlying assets to reduce uncertainty about the real world measurement or it may use the observable prices of traded derivatives to reduce uncertainty about the risk-neutral measurement. The use of derivatives, however, has drawbacks:

- Illiquidity regarding maturities, strikes and volume, or with different payoffs, making it difficult to properly define the hedging strategy.
- The market is incomplete.
- The sophisticated pricing models and strategies do not match the behavior of the underlying assets (see Cont (2006) in example 4.5, and Bakshi et al. (1997) used dynamic simulation hedging to find that the most sophisticated model may not be the most effective tool for hedging).
- Different parameters for several models can provide the same solution (see Cont (2006) in example 4.4).

For the vast majority of assets, a price cannot be directly observed and has to be inferred from observable prices of benchmark instruments. This is typically the case for financial derivatives whose prices are related to various features of their underlying assets. This process is known as marking to model (see Rebonato 2003) and involves both a mathematical algorithm and subjective aspects, exposing the process to a variety of errors. The solution is then bounded but not unique and the question is how it is distributed in this interval.

Frey and Sin (2001) points out that, in practice, it may not be possible to determine a finite upper bound on the volatility process, although if the model satisfies the Hadamard means criteria (i.e., for all admissible data a unique solution exists that depends continuously on the data) then we call the process of approximating an ill-posed problem with a well-posed problem regularization. A common way of addressing the potential non-existence of a solution is to replace equation with a minimization (least-squares) problem.

In some cases, Lindley (2006) argues that it is as important to be able to measure the uncertainty of a model parameter as it is to find the model parameter. One method of measuring the potential error is precisely to consider the model parameters as a random variable, assigning them a probability distribution.

Deryabin (2012) looks at upper and lower bounds on coherent model risk measurements. He defines parsimonious linear bounds for the model calibration uncertainty that depend only on the stochastic dynamics specified by the model and its calibration and not on a particular choice of modeling parameters. But this is only true for derivatives with liquidity market price and needs bid-ask prices or benchmark

Sibbertsen et al. (2008) reviews the calibration of financial risk models and defines model risk as the discrepancy between the implemented data generating process and the data observed, i.e., the model risk has three levels: selection of probability distribution; econometric models (see Carcano 2009) and tests, and the estimate of parameters. Boucher et al. (2014) estimates model risk as a biased VaR estimate with respect to true VaR but only analyzes it for a known econometric DGP (GARCH t-Student); Danielsson et al. (2016) follows this analysis but includes more econometric models. Brotcke (2018) uses a logit model to measure the model robustness index to control how the risk estimate methodologies behave in different events. Model risk describes the risk that the model used adjust the market data incorrectly.

Some studies focus on model uncertainty and model risk but do not calculate a capital risk charge as a result of measuring unexpected losses (Hénaff and Martini 2011; Cohort et al. 2013). Only a few papers, however, are concerned with determining capital requirements for model risk (Merton 1974; Elices and Giménez 2014; Glasserman and Xu 2014; Boucher et al. 2014; Breuer and Csiszar 2016; Detering and Packham 2016).

Among the different causes of model risk, we study the misspecification of the underlying stochastic process and calibration and revision of estimated parameters, in line with the definition of incorrect model risk in Derman (1996) or value approach. As Morini (2011) indicates, the risk model is the significant difference between mark-to-model value and price market or price approach. However, as Kato and Yoshiba (2000) points out, this includes the risk result of the discretization error and other approaches. Bignozzi and Tsanakas (2016) studies the impact of parameter uncertainty on capital adequacy for a given risk measurement and capital estimate procedure then proposes modified capital estimate procedures based on parametric bootstrapping and on predictive distributions. Feng et al. (2021) uses relative entropy to measure calibration error and Black-Scholes model risk due to recalibration. In line with the Bayesian proposal of Sibbertsen et al. (2008) Bignozzi and Tsanakas (2016) and Runaru and Zheng (2017), but using Monte Carlo approaches, we calculate the expected risk measure by taking the average of all candidate models. The main shortcoming of this approach consists of the difficulties to measure the probability of each model a priori. We propose a linear possibility by least-squares as shown in Frey and Sin (2001). Thus, our contribution consists of providing a methodological response to the regulatory requirements on the control of model risk; specifically, we propose a simulation algorithm and a least squares estimation that allows us to determine the model that best fits the data and, once the parameters have been estimated recursively, to estimate both market risk (with the mean value of these parameters) and model risk (with extreme values of these parameters).

The rest of the paper is structured as follows. Section II proposes the methodology. Section III shows experimental and market data results and Section IV offers concluding remarks.

Methodology to measure the model risk

Base Line model risk

Historically, the stochastic processes have been used by several authors (Bachelier 1990; Osborne 1959; Samuelson 1965) to explain the behavior of the return series of financial assets.

In modern finance it is common to use the so-called Geometric Brownian Motion (GBM) to define the behavior of return-on-asset prices. These processes are characterized by two components, drift (μ) and diffusion (σ). While the first is the trend, the second shows the volatility which follows the trend. Both are unknown parameters and cause model risk. If S_t is the natural logarithm of the price at each instant of time t, the behavior of this variable in discrete time is given by (1), from applying the *Itô* Lemma to the process in continuous time:

$$\Delta S_t = (\mu - 0.5\sigma^2)\Delta t + \sigma \Delta W_{S,t} \tag{1}$$

where W_t is the Wiener process defined in (2).

$$\Delta W_{S,t} \sim \mathbf{N}(0, \Delta t) \sim \Delta t\mathbf{N}(\mathbf{0}, \mathbf{1}) \rightarrow \Delta W_{S,t} = \sqrt{(}\Delta t)\epsilon_{S,t} \tag{2}$$

$\mathbf{N}$ is standard cumulative normal distribution and $\epsilon_{S,t}$ is a random standard normal variable.

We define a stochastic model with constant drift and diffusion as Base Line Model in (3).

$$\Delta S_t = (\mu - 0.5\sigma^2)\Delta t + \sigma \sqrt{\Delta t}\epsilon_{S,t} \tag{3}$$

Our aim is to determine the parameters that define the behavior of these processes to find out which is the implicit market model. The financial literature offers two approaches: market and equilibrium models. While the former are based on the principles that inspired the assessment of contingencies proposed by Black and Scholes (1973) and Merton (1974), using self-financing portfolios and free arbitrage opportunity prices, the latter raises questions about equilibrium price independently of market prices. In the first case, the usual way to calibrate the parameters is to solve the stochastic partial differential equation for a hedged portfolio that cannot outperform the risk-free rate chosen as *numerarie*. However, as already pointed out, if the market is incomplete or shows liquidity problems then we need to use another procedure. For these cases, we propose a least-squares Monte Carlo methodology in line with Avellaneda et al. (2000), Longstaff and Schwartz (2001), and Glasserman and Xu (2014). Avellaneda et al. (2000) use a *non-uniformly weighted* Monte Carlo simulations, but since our objective is to determine the model that best fits the behavior of the returns of the assets traded in the market and, unlike (Avellaneda et al. 2000) we do not assume that both the underlying asset and the derivative are traded), our proposal uses an *uniformly weighted* simulation. And unlike (Glasserman and Xu 2014), our objective is not to estimate the risk model as sensitivity to changes in the probability distribution, but to identify the process followed by asset returns and sensitivity to the parameters that define it. While Glasserman and Xu (2014) proposal assumes the ability to generate values (simulate) from any path of stochastic processes, we define these simulations as a way to estimate the market and model risks simultaneously.

This has two major implications:

- The estimate uses a sample of prices and not the whole population. Both Avellaneda et al. (2000) and our estimates show sampling errors (the finite sample effect).

- The random number simulation also experiences the same effect since the unknown parameters that cause the original shocks are unobservable.

We define $\epsilon_{S,t}$ and $\epsilon^*_{S,t}$ as real and simulated standard normal random shocks, respectively. Since the real shock is not observable because we do not know the value of model parameters, we use a simulated shock to estimate the model. If Q is percentile rank function, then $\epsilon^*_{S,t} = \mathbf{N}^{-1}(Q_{S,t})$ with the first and second sample moments (mean and variance) equal to 0 and 1, respectively. According to (3), we express the linear relations between return rates and each of these random numbers as (4).

$$\begin{aligned} \Delta S_t &= (\mu - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\epsilon_{S,t} \\ \hat{\Delta S}_t &= a + b\epsilon^*_{S,t} \\ e_t &= \Delta S_t - \hat{\Delta S}_t \end{aligned} \tag{4}$$

where $\mathbf{E}(\epsilon^*_{S,t}) = 0$, $\mathbf{E}(\epsilon^*_{S,t}, \Delta S_t) = \rho_{\epsilon_S,\Delta S} = 1$ and, e_t is the error to be minimized ($min. \sum_{t=1}^{T} e_t^2$). For a return sample of size T and τ sample for estimating, the basic procedure proposal is, first, to estimate percentile rank of observed return and generate the corresponding T normal standard random numbers ($\epsilon^*_{S,t}$). Secondly, for $j = 1$ to $T - \tau$, estimate the diffusion and drift parameters in (4) by least squares:

$$\begin{aligned} \hat{\sigma}_j &= \frac{b}{\sqrt{\Delta t}} \\ \hat{\mu}_j &= \frac{a}{\Delta t} + 0.5\hat{\sigma}_j^2 \end{aligned} \tag{5}$$

Finally, we define the Value at Risk (VaR[1]) and Model at Risk (MaR) for an α confidence interval and time horizon Δt in (6):

$$\begin{aligned} P_{T+\Delta t} &= \exp(S_T) \\ VaR_{(\Delta t,\alpha,\hat{\sigma},\hat{\mu})} &= \{\exp[(\hat{\mu} - 0.5\hat{\sigma}^2)\Delta t + \hat{\sigma}\sqrt{\Delta t}\epsilon_\alpha] - 1\}P_T \\ MaR_{(\Delta t,\sigma_\alpha,\mu_\alpha)} &= \{\exp[(\mu_\alpha - 0.5\sigma_\alpha^2)\Delta t + \sigma_\alpha\sqrt{\Delta t}\epsilon_\alpha] - 1\}P_T - VaR_{(\Delta t,\alpha,\hat{\sigma},\hat{\mu})} \end{aligned} \tag{6}$$

where P_T is the current asset price, ϵ_α is the value for the αth percentile on a standard normal distribution and, μ_α and σ_α are the values for αth percentile among the estimaties.

Generalized model risk

Under the characteristics of stochastic models, there are different proposals about the model components (drift and diffusion) and from these a diversity of models arises. We define the following models with their corresponding empirical expressions which we use to estimate the parameters for each model:

[1] We may also use other risk measures such as Conditional-VaR or Shortfall

- Model-0. Drift and diffusion are constants as (3) or Base Line Model and, we estimate their unkown parameters by (5).
- Model-I. Reversion mean and constant diffusion (7). This model is an *Ornstein-Uhlenbeck* process for the stochastic variable $S_t = \ln P_t$. It means that there is a reversion force on S_t pushing towards an equilibrium level (μ), γ is the velocity of the reversion and $\frac{\ln 2}{\gamma}$ is the *half-life* or the expected time for S_t to reach the half way point to the equilibrium level (a revision of this type of models is Chan et al. (1992)). To avoid discretization errors (i.e. using Euler or Milstein approximations), we use the correct discrete time format for this process for both forms in continuous time, i.e., additive and geometric. In (7), we show the exact (valid for large Δt) discrete time expression (see Dixit et al. 1994, p. 76) to simulate and the llinear model to estimate unkown parameters:

$$\begin{aligned} S_t &= \mu[1-\exp(-\gamma\Delta t)] - [1-\exp(-2\gamma\Delta t)]\frac{\sigma^2}{4\gamma} + \sigma\sqrt{\frac{1-\exp(-2\gamma\Delta t)}{2\gamma}}\epsilon_{S,t} \\ &\quad + S_{t-1}\exp(-\gamma\Delta t) \\ \hat{S}_t &= a + b\epsilon^*_{S,t} + cS_{t-1} \\ e_t &= S_t - \hat{S}_t \end{aligned} \tag{7}$$

For *additive* process $\mu = \ln \bar{P}$, where $\bar{P}$ is the long-run equilibrium price and the estimation of the parameters is:

$$\begin{aligned} \hat{\gamma} &= -\frac{\ln c}{\Delta t} \\ \hat{\sigma} &= \frac{b}{\sqrt{\Delta t}}\sqrt{-\frac{2\ln c}{1-c^2}} \\ \hat{\mu} &= \frac{a + [1-\exp(-2\gamma\Delta t)]\frac{\sigma^2}{4\gamma}}{1-c} \end{aligned} \tag{8}$$

For the *geometric* process (see Schwartz 1997) $\mu^* = \mu - \frac{\sigma^2}{2\gamma}$ so for (8) we need only re-estimate the equilibrium level parameter as:

$$\hat{\mu} = \frac{a + [1-\exp(-2\gamma\Delta t)]\frac{\sigma^2}{4\gamma}}{1-c} + \frac{b}{1-c^2} \tag{9}$$

It also applies to other cases like in Ho and Lee (1986), where the interes rate (r) shows time dependent drift and constant diffusion. In continuous time the Ho and Lee (1986) model is: $dr_t = (\theta_t - \gamma r_{t-1})dt + \sigma dW_t = \theta_t + x_t$, where θ_t is a deterministic component that depends on market forward rates and x_t is the stochastic component or *Ornstein-Uhlenbeck* process with null drift as: $dx_t = -\gamma r_{t-1}dt + \sigma dW_t$. For $\theta_0 = r_0$ and $x_0 = 0$, we use the same procedure to estimate the unkown parameters (see Chan et al. 2015, pp. 167–170).

- Model-II. Stochastic drift and constant diffusion. The behavior of asset price is usually defined in neutral risk terms, so the drift depends on the risk-free interest rate. Both the asset price and the interest rate are stochastic. For example, we define the behavior of the interest rate as a variant of *Model-I*, while the asset price follows a *GBM* or *Model*-0. The interest rate discretization model, e.g. with κ parameter to avoid negative values (see Cox et al. 1985 where $\kappa = 0.5$) is:

$$\begin{aligned} r_t &= \mu_r[1-\exp(-\gamma_r \Delta t)] - [1-\exp(-2\gamma r \Delta t)]\frac{\sigma_r^2}{4\gamma_r} + r_{t-1}^{\kappa}\sigma_r\sqrt{\frac{1-\exp(-2\gamma\Delta t)}{2\gamma}}\epsilon_{S,t} \\ &\quad + r_{t-1}\exp(-\gamma_r \Delta t) \\ \hat{r}_t &= a_r + b_r\epsilon^*_{r,t} r^{\kappa}_{t-1} + c_r r_{t-1} \\ e_{r,t} &= r_t - \hat{r}_t \end{aligned} \tag{10}$$

were the unkown parameters are estimated as (8). Since return assets are correlated ($\rho_{r,S}$) to interest rate movements, we apply Cholesky decomposition (see Chan et al. 2015, pp. 165–166):

$$\begin{aligned} \Delta y_t &= \Delta S_t - r_t = -0.5\sigma^2_{\Delta S}\Delta t + \sigma_{\Delta S}\sqrt{\Delta t}\epsilon_{S,t} \\ \hat{\Delta y}_t &= a_S + b_S\epsilon^*_{S,t} + c_S\epsilon^*_{r,t} \\ e_{S,t} &= \Delta y_t - \hat{\Delta y}_t \end{aligned} \tag{11}$$

And the searched parameters are:

$$\begin{aligned} \hat{\rho}_{r,S} &= \mathbf{sgn}(\frac{b_S}{c_S})\sqrt{\frac{b_S^2}{b_S^2 + c_S^2}} \\ \hat{\sigma}_S &= \frac{c_S}{\hat{\rho}_{r,S}\sqrt{\Delta t}} \end{aligned} \tag{12}$$

The model restriction is $\frac{2\gamma_r\mu_r}{\sigma_r^2} \geq 1$ or the Feller condition. Applying this condition on (10) results in:

$$\begin{aligned} \frac{2\gamma_r\mu_r}{\sigma_r^2} &= \frac{2\frac{c_r}{\Delta t}\frac{0.5a_r + b_r^2}{c_r}}{\frac{2b_r}{\sqrt{\Delta t}}} \\ &= \frac{0.5a_r + b_r^2}{2b_r^2} \Rightarrow \frac{a_r}{2b_r} \geq 1 \end{aligned} \tag{13}$$

- Model-III. Jump diffusion like in Merton (1976) or Ball and Torous (1983), but in this case we distinguish between jumps up (+) and down (−):

$$\begin{aligned}
\Delta S_t &= (\mu - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\epsilon_{S,t} + J_t^+ + J_t^- \\
J_t^+ &= J_{t-1}^+ + \mathbf{I}_t^+(\eta^+ + \delta\epsilon_{J,t}^+) \\
J_t^- &= J_{t-1}^- + \mathbf{I}_t^-(\eta^- + \delta\epsilon_{J,t}^-) \\
\mathbf{I}_t^+ &= \begin{cases} 1 & \text{if } u_t < \lambda^+\Delta t \\ 0 & \text{otherwise} \end{cases} \\
\mathbf{I}_t^- &= \begin{cases} 1 & \text{if } u_t > 1 - \lambda^-\Delta t \\ 0 & \text{otherwise} \end{cases} \\
u_t &\sim \mathbf{U}(0,1) \\
\epsilon_{S,t}, \epsilon_{J,t}^+, \epsilon_{J,t}^- &\sim \mathbf{N}(0,1) \\
\mathbf{E}(\epsilon_{S,t}, \epsilon_{J,t}^+) &= 0 \\
\mathbf{E}(\epsilon_{S,t}, \epsilon_{J,t}^-) &= 0 \\
\mathbf{E}(\epsilon_{J,t}^+, \epsilon_{J,t}^-) &= 0
\end{aligned} \tag{14}$$

where $\mathbf{U}$ is an uniform distribution, $\mathbf{N}$ is a normal standard distribution, $\lambda^{+,-}$ is the annualized frecuency of jumps up and down and the size of jumps follows a $\mathbf{N}(\eta^{+,-}, \delta^{2,+,-})$. Note that shocks $(\epsilon_S, \epsilon_J^{+,-})$ are i.i.d. Then we estimate the unkown parameters in two-steps:

- First, we use (4) and (5) to estimate μ and σ. We calculate: $e_t = \Delta S_t - [(\hat{\mu} - 0.5\hat{\sigma}^2)\Delta t + \hat{\sigma}\sqrt{\Delta t}\epsilon_{S,t}^*]$. If $x_t = \frac{z_t - mean_e}{desv_e}$ is the standardized residuals and p_{x_t} are the correponding percentiles then we estimate the annualized jump frecuencies as:

$$\begin{aligned}
\hat{\lambda}^+ &= min(p_{e_t} | \forall x_t > C)\Delta t^{-1} \\
\hat{\lambda}^- &= max(p_{e_t} | \forall x_t < C)\Delta t^{-1}
\end{aligned} \tag{15}$$

  Note that we select only higher values of C for jumps up and less than $-C$ to jump down (if we cannot distinguish between jumps up and down, then we would take the values of x in absolute value and we would select the highest values of C).
- Second, once the jump frequencies have been estimated, we proceed to estimate the jump size parameters by:

  1. Simulating uniform random numbers (u), as many as observations (τ).
  2. Creating a matrix D_{Nx3}, where each row r_D corresponds to each simulated variable u and is defined as: If u is less than $\hat{\lambda}^+\Delta t$ then $r_D = (1, 0, 0)$; if higher than 1-$\hat{\lambda}^-\Delta t$ then $r_D = (0, 0, 1)$; otherwise $r_D = (0, 0, 0)$.
  3. Generating A as a vector $(Nx1)$ of standard normal random numbers and appliying the Hadamard product. In such a case: $A^* = A \circ D'$.
  4. Generating B as a vector $(Nx1)$ of ones and appliying the Hadamard product to obtain: $B^* = B \circ D'$.

5. If Z is a vector ($Nx1$) of standardized residuals then generating $Y = Z \circ D'$.
6. Concatenating matrix A^* and B^* we obtain: $X = B^* \parallel A^*$ and then, $\hat{Y} = \theta \cdot X$.
7. Estimating parameters $\theta = (\eta^{+,-}, \delta^{+,-})$ by least squares of errors: $e^2 = (Y - \hat{Y})^2$.
8. Repite $T - \tau$ times.

- Model-IV. Stochastic volatily model. In this case, we consider two approachs:

1. Volatility without mean reversion like in Hull and White (1987). The discretized stochastic processes for asset return and volatility are (see equation 11 in Hull and White 1987):

$$\begin{aligned} S_t &= S_{t-1} \exp[(\mu - 0.5\sigma_t^2)\,\Delta t + \sigma_t \sqrt{\Delta t}\epsilon_{S,t}] \\ \sigma_t &= \sigma_{t-1} \exp[(\mu_\sigma - 0.5\nu^2)\Delta t + \nu\sqrt{\Delta t} z_{\sigma,t}] \\ z_{\sigma,t} &= \rho_{S,\sigma}\epsilon_{S,t} + \sqrt{(1-\rho^2_{S,\sigma})}\epsilon_{\sigma,t} \end{aligned} \tag{16}$$

where $\mathbf{E}(\epsilon_{S,t}, \epsilon_{\sigma,t}) = 0$.

2. Volatility with mean reversion like in Scott (1987), Stein and Stein (1991) and Heston (1993) among others, where the corresponding discretized stochastic processes are (see Scott 1987 for *Ornstein-Uhlenbeck* process for volatility):

$$\begin{aligned} S_t &= S_{t-1} \exp[(\mu - 0.5\sigma_t^2)\Delta t + \sigma_t \sqrt{\Delta t}\epsilon_{S,t}] \\ \sigma_t &= \sigma_{t-1} \exp(-\gamma_\sigma \Delta t) + \mu_\sigma[1 - \exp(-\gamma_\sigma \Delta t)] + \nu\sqrt{\frac{1-\exp(-2\gamma_\sigma\Delta t)}{2\gamma_\sigma}} z_{\sigma,t} \end{aligned} \tag{17}$$

The expression (17), as with the expression (10), has a similar constraint (*Feller* condition, see (13)). As volatility is not a variable traded directly on the marketplace, we propose a methodology in line with Broze et al. (1998) and Gouriéoux and Monfort (1996), but it also may be applied in a thinly traded market. Fiorentini et al. (2002) uses a time-series approach to estimate the parameters of stochastic volatility model. This indirect inference, using the *NAGARCH* model, has a drawback. Our methodology simulates shocks which are independent of the discretization error, while the use of an auxiliary model assumes that residuals show both together (true shock and discretization error), with the corresponding effect on the estimated parameters. Additionally, as noted by Gouriéoux and Monfort (1996), to use auxiliary models the process must be simulated, which is questionable at least in the case of NAGARCH as a discretization approach to the Heston model. But this happens in other models like *GARCH* (see Heston and Nandi (2000)) as well. The simulation of stochastic volatility models was initially performed by the usual *Euler* or *Milstein* discretization, then more sophisticated methods that converge faster were used, avoiding potential negative values, such as volatility transformed or quadratic exponential scheme (see Gatheral 2006; Kahl and Jäckel 2006; Andersen 2008; Lord et al. 2010; Zhu 2010). A review is available at Van Haastrecht and Pelsser (2010) and Rouah (2015). Since *Feller* condition is set in continuous time, and the simulation and observations are at discrete time, the first two approaches are: *Full truncation* scheme, where the simulated volatility (σ_t) is $\sigma_t = max(0, \sigma_t)$, but it generates null values for volatility; and a *Reflection* scheme defined as $\sigma_t = |\sigma_t|$, although it generates high positive values when the simulated volatility value is negative. Because of the problems in these simple approaches, some authors choose to simulate the square root (see Zhu 2010). In this case the drawback is that the mean level of volatility becomes stochastic, since it depends on the previous value of the volatility. Others simulate the logarithm of volatility to avoid negative values, or use other probability distributions (*Chi-squared*), which increase the number of parameters to estimate, but convergence is conditioned by the value of certain parameters, e.g. the volatility of volatility in Kahl and Jäckel (2006) ś scheme; or we have to simulate alternative distribution probabilities (e.g. *Chi-squared* and *Gamma*) lilke (Broadie and Kaya 2006). Additionally, the parameters estimate requires a liquid options market, and the results obtained by different calibration methods are not always similar (see Rouah 2015, pp. 123–124). In addition, for both the simulation and the estimate we must add one more parameter to the model, the initial value of volatility, which is not observable, unlike prices or returns. Our proposal does not add more parameters than the strictly defined stochastic processes of return and volatility and they are available, as we initially noted, for thinly traded markets where there are no options or which are illiquid. We propose the following substitution in discrete time:

$$\sigma_t^* = \frac{\Delta S_t}{\sqrt{\Delta t}\epsilon_{S,t}} - \frac{\mu\sqrt{\Delta t}}{\epsilon_{S,t}} \tag{18}$$

The expression (18) is a proxy of:

$$\begin{aligned} \Delta S_t &= (\mu - 0.5\sigma_t^2)\Delta t + \sigma_t\sqrt{\Delta t}\epsilon_{S,t} \\ \sigma_t &= \frac{\epsilon_{S,t}}{\sqrt{\Delta t}} \pm \sqrt{\frac{\epsilon_{S,t}^2}{\Delta t} + 2\mu - 2\frac{\Delta S_t}{\Delta t}} \end{aligned} \tag{19}$$

From (17) and (18), we obtain the following expression for volatility without the mean reversion model:

$$\begin{aligned} \frac{\Delta S_t}{\sqrt{\Delta t}\epsilon_{S,t}} - \frac{\Delta S_{t-1}}{\sqrt{\Delta t}\epsilon_{S,t-1}} = (\mu_\sigma - 0.5\nu^2)\Delta t + \mu\sqrt{\Delta t}\left(\frac{1}{\epsilon_{S,t}} - \frac{1}{\epsilon_{S,t-1}}\right) + \\ \nu\sqrt{\Delta t}\rho_{S,\sigma}\epsilon_{S,t} + \nu\sqrt{\Delta t}\sqrt{1-\rho_{S,\sigma}^2}\epsilon_{\sigma,t} \end{aligned} \tag{20}$$

And (17) and (18) show this expression for volatility with the mean reversion model:

$$\frac{\Delta S_t}{\sqrt{\Delta t}\epsilon_{S,t}} = \frac{\mu\sqrt{\Delta t}}{\epsilon_{S,t}} + \mu_\sigma[1-\exp(-\gamma_\sigma\Delta t)] + \exp(-\gamma_\sigma\Delta t)\left(\frac{\Delta S_{t-1}}{\sqrt{\Delta t}\epsilon_{S,t-1}} - \frac{\mu\sqrt{\Delta t}}{\epsilon_{S,t-1}}\right)+ \nu\sqrt{\frac{1-\exp(-2\gamma_\sigma\Delta t)}{2\gamma_\sigma}}\rho_{S,\sigma}\epsilon_{S,t} + \nu\sqrt{\frac{1-\exp(-2\gamma_\sigma\Delta t)}{2\gamma_\sigma}}\sqrt{(1-\rho^2_{S,\sigma})}\epsilon_{\sigma,t} \tag{21}$$

If we define: $x_{0,t}=1$, $x_{1,t}=(\frac{1}{\epsilon_{S,t}}-\frac{1}{\epsilon_{S,t-1}})$, $x_{2,t}=\epsilon_{S,t}$, $x_{3,t}=\epsilon_{\sigma,t}$, $y_t=\frac{\Delta S_t}{\sqrt{\Delta t}\epsilon_{S,t}}$ and $\Delta y_t = y_t - y_{t-1}$, then we rewrite the expressions (20) and (21) as:

$$\begin{aligned} \textit{No mean reversion:}\quad \hat{\Delta y}_t &= a_0x_{0,t} + a_1x_{1,t} + a_2x_{2,t} + a_3x_{3,t} \\ e_{1,t} &= \Delta y_t - \hat{\Delta y}_t \\ \textit{Mean reversion:}\quad \hat{y}_t &= b_0x_{0,t} + b_1x_{1,t} + b_2x_{2,t} + b_3x_{3,t} + cy_{t-1} \\ e_{2,t} &= y_t - \hat{y}_t \end{aligned} \tag{22}$$

where for both models the estimate of unknown parameters is:

$$\begin{aligned} \hat{\mu} &= \begin{cases} \frac{a_1}{\sqrt{\Delta t}} \\ \frac{b_1}{\sqrt{\Delta t}} \end{cases} \\ \hat{\rho}_{S,\sigma} &= \begin{cases} \mathbf{sgn}(\frac{a_2}{a_3})\sqrt{\frac{a_2^2}{a_2^2+a_3^2}} \\ \mathbf{sgn}(\frac{b_2}{b_3})\sqrt{\frac{b_2^2}{b_2^2+b_3^2}} \end{cases} \\ \hat{\nu} &= \begin{cases} \frac{a_2}{\hat{\rho}_{S,\sigma}\sqrt{\Delta t}} \\ \frac{b_2}{\hat{\rho}_{S,\sigma}\sqrt{\frac{1-\exp(-2\hat{\gamma}_\sigma\Delta t)}{2\hat{\gamma}_\sigma}}} \end{cases} \\ \hat{\mu}_\sigma &= \begin{cases} \frac{a_0}{\Delta t} + 0.5\hat{\nu}^2 \\ \frac{b_0}{1-c} \end{cases} \\ \hat{\gamma}_\sigma &= -\frac{\ln c}{\Delta t} \end{aligned} \tag{23}$$

- Model-V. Drift and diffusion are stochastic, and both follow an mean reversion process:

$$
\begin{aligned}
\Delta S_t &= (r_t - 0.5\sigma_t^2)\Delta t + \sigma_t \sqrt{\Delta t}\epsilon_{S,t} \\
r_t &= \mu_r[1 - \exp(-\gamma_r \Delta t)] + r_{t-1} \exp(-\gamma_r \Delta t) + \sigma_r \sqrt{\frac{1 - \exp(-2\gamma_r \Delta t)}{2\gamma_r}} z_{r,t} \\
\sigma_t &= \mu_\sigma[1 - \exp(-\gamma_\sigma \Delta t)] + \sigma_{t-1} \exp(-\gamma_\sigma \Delta t) + \nu \sqrt{\frac{1 - \exp(-2\gamma_\sigma \Delta t)}{2\gamma_\sigma}} z_{\sigma,t} \\
z_{r,t} &= \rho_{r,S}\epsilon_{S,t} + \sqrt{1 - \rho_{r,S}^2}\epsilon_{r,t} \\
z_{\sigma,t} &= \rho_{S,\sigma}\epsilon_{S,t} + \frac{\rho_{r,\sigma} - \rho_{r,S}\rho_{S,\sigma}}{\sqrt{1 - \rho_{r,S}^2}}\epsilon_{r,t} + \sqrt{1 - \frac{\rho_{S,\sigma}^2 + \rho_{r,\sigma}^2 - 2\rho_{S,\sigma}\rho_{r,S}\rho_{r,\sigma}}{1 - \rho_{r,S}^2}}\epsilon_{\sigma,t}
\end{aligned} \tag{24}
$$

Then, as in (18), we define the following volatility proxy:

$$
\sigma_t^* = \frac{\Delta S_t}{\sqrt{\Delta t}\epsilon_{S,t}} - \frac{r_t \sqrt{\Delta t}}{\epsilon_{S,t}} \tag{25}
$$

And we obtain the follow expression:

$$
\begin{aligned}
\hat{r}_t &= a_0 + a_1 \cdot r_{t-1} + a_2 \cdot \epsilon_{S,t} + a_3 \cdot \epsilon_{r,t} \\
\hat{\sigma}_t^* &= b_0 + b_1 \cdot \hat{\sigma}_{t-1} + b_2 \cdot \epsilon_{S,t} + b_3 \cdot \epsilon_{r,t} + b_4 \cdot \epsilon_{\sigma,t} \\
e_{1,t} &= r_t - \hat{r}_t \\
e_{2,t} &= \sigma_t^* - \hat{\sigma}_t
\end{aligned} \tag{26}
$$

And if we operate on expression (26), we obtain the unknown parameters:

$$
\begin{aligned}
\hat{\gamma}_r &= -\frac{\ln(a_1)}{\Delta t} \\
\hat{\gamma}_\sigma &= -\frac{\ln(b_1)}{\Delta t} \\
\hat{\mu}_r &= \frac{a_0}{1 - a_1} \\
\hat{\mu}_\sigma &= \frac{b_0}{1 - b_1} \\
\hat{\rho}_{r,S} &= \mathbf{sgn}\left(\frac{a_2}{a_3}\right)\sqrt{\frac{a_2^2}{a_2^2 + a_3^2}} \\
\hat{\sigma}_r &= \frac{a_2}{\hat{\rho}_{r,S}\sqrt{\Delta t}} \\
\hat{\rho}_{S,\sigma} &= \sqrt{\frac{b_2^2(1 - \hat{\rho}_{r,S}^2)}{b_2^2 + b_3^2 + b_4^2 + (b_2^2 - b_3^2 - 3b_4^2)\hat{\rho}_{r,S}^2 + (b_2 b_3^2 - 2b_4^2)\hat{\rho}_{r,S}\sqrt{(1 - \hat{\rho}_{r,S}^2)}}} \\
\hat{\rho}_{r,\sigma} &= \frac{b_3}{b_2}\hat{\rho}_{S,\sigma}\sqrt{(1 - \hat{\rho}_{r,S}^2)} + \rho_{S,\sigma}\hat{\rho}_{r,S} \\
\hat{v} &= \frac{b_2}{\hat{\rho}_{S,\sigma}\sqrt{\frac{1-\exp(-2\hat{\gamma}_\sigma \Delta t)}{2\hat{\gamma}_\sigma}}}
\end{aligned}
\tag{27}
$$

- Model-VI. Constant drift and jump-diffusion as in Bates (1996) or Pan (2002), among others. This model is a mix of Model-III (14) and Model-IV (16):

$$
\begin{aligned}
S_t &= S_{t-1} \exp[(\mu - S_t - 0.5\sigma_t^2)\Delta t + \sigma_t \sqrt{\Delta t}\epsilon_{S,t}] + J_t^+ + J_t^- \\
\sigma_t &= \sigma_{t-1} \exp(-\gamma_\sigma \Delta t) + \mu_\sigma[1 - \exp(-\gamma_\sigma \Delta t)] + \nu \sqrt{\frac{1 - \exp(-2\gamma_\sigma \Delta t)}{2\gamma_\sigma}} z_{\sigma,t} \\
J_t^+ &= J_{t-1}^+ + \mathbf{I}_t^+(\eta^+ + \delta\epsilon_{J,t}^+) \\
J_t^- &= J_{t-1}^- + \mathbf{I}_t^-(\eta^- + \delta\epsilon_{J,t}^-) \\
\mathbf{I}_t^+ &= \begin{cases} 1 & \text{if } u_t < \lambda^+ \Delta t \\ 0 & \text{otherwise} \end{cases} \\
\mathbf{I}_t^- &= \begin{cases} 1 & \text{if } u_t > 1 - \lambda^- \Delta t \\ 0 & \text{otherwise} \end{cases} \\
u_t &\sim \mathbf{U}(0, 1) \\
z_{\sigma,t} &= \rho_{S,\sigma}\epsilon_{S,t} + \sqrt{(1 - \rho_{S,\sigma}^2)}\epsilon_{\sigma,t} \\
\epsilon_{S,t}, \epsilon_{\sigma,t}, \epsilon_{J,t}^+, \epsilon_{J,t}^- &\sim \mathbf{N}(0, 1) \\
\mathbf{E}(\epsilon_{S,t}, \epsilon_{\sigma,t}) = 0 \quad & \mathbf{E}(\epsilon_{S,t}, \epsilon_{J,t}^+) = 0 \\
\mathbf{E}(\epsilon_{S,t}, \epsilon_{J,t}^-) = 0 \quad & \mathbf{E}(\epsilon_{\sigma,t}, \epsilon_{J,t}^+) = 0 \\
\mathbf{E}(\epsilon_{\sigma,t}, \epsilon_{J,t}^-) = 0 \quad & \mathbf{E}(\epsilon_{J,t}^+, \epsilon_{J,t}^-) = 0
\end{aligned}
\tag{28}
$$

In this case, first we estimate the parameters without jumps in accordance with Model-IV and then, given the independence between the shocks, we estimate the parameters of the jump (see Model-III estimate procedure).

Results

This section contains two empirical parts. The first consists of an experimental analysis in which the robustness of our methodology is contrasted using simulated series for the different models analyzed, so that we test whether the estimated parameters are statistically equal to those used in the simulation (real). In the second part, the proposed methodology is applied to a real series of a financial asset to analyze which model shows the best fit and check the risk estimates.

Experimental analysis

First, we simulate 10,000 time series for each model using VBA software, then we estimate the parameters with recursive regression initializing the estimate with 50% of the sample. We also use software to make the estimates (Ox), so that the random number simulators are different. Table 1 shows the results by quantiles of the estimated parameters by least squares and we define the goodness of fit of each model as $R^2 = 1 - \frac{\sum_t e_t^2}{\sum_t y_t^2}$, where e is the least squares estimation error and y is the simulated value.

The results of Table 1 show that the goodness of fit of the models is between a minimum of 90.32% and a maximum of 99.84%, which means that the proposed

Table 1 Parameters of recursive estimates

Model	Quantile	R^2	S			r				σ				
			a	*b*	*c*	*a*	*b*	*c*	*d*	b_0	b_1	b_2	b_3	*c*
0	min	96.46%	− 0.001	0.012										
	Q1	97.63%	0.001	0.012										
	Q2	98.76%	0.002	0.013										
	Q3	99.30%	0.002	0.013										
	max	99.84%	0.004	0.013										
1	min	91.81%	− 0.001	0.012	0.001									
	Q1	93.90%	− 0.001	0.013	0.002									
	Q2	94.92%	− 0.001	0.013	0.003									
	Q3	96.92%	− 0.001	0.013	0.004									
	max	98.93%	0.001	0.013	0.006									
2	min	93.05%	0.18	− 0.003	0.002	0.001	0.005	− 0.004						
	Q1	94.09%	0.191	− 0.002	0.002	0.002	0.005	− 0.003						
	Q2	95.02%	0.213	− 0.001	0.002	0.003	0.005	− 0.003						
	Q3	95.90%	0.208	− 0.001	0.003	0.048	0.006	− 0.003						
	max	97.29%	0.222	0.003	0.003	0.005	0.006	0.001						
3	min	90.65%	0.008	0.015										
	Q1	93.74%	0.002	0.015										
	Q2	96.99%	0.003	0.015										
	Q3	97.07%	0.001	0.016										
	max	98.30%	0.001	0.016										

Table 1 (continued)

Model	Quantile	R^2	S			r				σ				
			a	*b*	*c*	*a*	*b*	*c*	*d*	b_0	b_1	b_2	b_3	*c*
4	min	95.03%	− 0.004							0.083	− 0.001	− 0.004	− 0.005	− 0.263
	Q1	97.24%	− 0.001							0.084	0.004	− 0.001	− 0.002	− 0.255
	Q2	98.40%	− 0.001							0.086	0.006	− 0.001	0.002	0.004
	Q3	99.02%	0.004							0.087	0.008	0.003	0.003	0.27
	max	99.22%	0.001							0.089	0.009	0.008	0.011	0.302
5	min	90.32%				− 0.002	0.988	− 0.003	− 0.004	0.045	− 0.18	− 0.005	− 0.007	− 0.014
	Q1	92.84%				− 0.001	1.025	− 0.002	− 0.003	0.083	− 0.001	− 0.002	− 0.004	− 0.002
	Q2	96.69%				− 0.001	1.033	− 0.002	− 0.002	0.085	0.006	− 0.001	0.001	0.001
	Q3	97.94%				− 0.001	1.042	− 0.001	− 0.001	0.088	0.014	0.001	0.002	0.002
	max	99.46%				0.003	1.071	− 0.001	− 0.001	0.103	0.464	0.019	0.008	0.011
6	min	98.14%				− 0.001	− 0.012	0.008	− 0.006	0.194	0.002	− 0.018	− 0.01	− 0.026
	Q1	98.84%				− 0.002	− 0.009	0.011	− 0.003	0.197	0.003	− 0.012	0.001	− 0.005
	Q2	99.05%				0.002	− 0.008	0.014	− 0.002	0.201	0.004	− 0.005	0.003	0.004
	Q3	99.35%				0.003	− 0.006	0.014	− 0.001	0.204	0.005	− 0.003	0.008	0.005
	max	99.51%				0.006	− 0.004	0.017	0.002	0.213	0.006	− 0.001	0.011	0.026

For each stochastic variable (*S* or log− price, *r* or risk-free interest rate and σ o volatility), the estimated parameters for each of the estimations are shown according to the confidence level obtained from the total recursive calibrations performed

Table 2 Parameters of jumps

Quantile	$jump^{+}$		$jump^{-}$	
	a	*b*	*a*	*b*
Model-III				
min	0.046	0.037	– 0.04	0.033
Q1	0.05	0.05	– 0.038	0.046
Q2	0.059	0.054	– 0.037	0.057
Q3	0.069	0.082	– 0.032	0.064
max	0.081	0.09	– 0.019	0.076
Model-VI				
min	0.034	0.041	– 0.003	0.027
Q1	0.044	0.057	– 0.064	0.041
Q2	0.057	0.065	– 0.073	0.062
Q3	0.064	0.086	– 0.094	0.889
max	0.084	0.102	– 0.137	0.138

methodology shows a high level of explanation of the stochastic models in discrete time. Table 2 shows the estimated parameters for models with jumps by least squares.

Finally, Table 3, we show the mean value of parameters for each stochastic model from transforming the previous estimated parameters by least squares. We also estimate the standard deviation for these transformed parameters and then test if we reject the null hypothesis that the mean value of the estimated parameters is equal to the real value used to generate the simulations.

Note that the individual hypothesis of equality between the real parameter (θ) used for the simulation and the estimated parameter ($\hat{\theta}$) is accepted in all cases.

Market data analysis

We apply the methodology to a sample that consists of daily market prices of Bitcoin, Brent, Eurostoxx-50 and Gold from January 1, 2014 to December 31, 2020. These data are obtained from Bloomberg.

First, we select the best model for each asset depending on adjusted R^2. Table 4 show the results.

Next, we simultaneously estimate market and model risk. Once the model of each asset (highest adjusted R^2) has been selected, we divide the sample into two parts: the sample to estimate parameters (from January 1, 2014 to December 31, 2018) and the sample to calculate and compare risks (from January 1, 2019 to December 31, 2020). For each date of the second sample, we apply a recursive procedure in three stages: first we estimate the model's parameters with the information available to date (the first estimate is with the first part of the sample, from January 1, 2014 to December 31, 2018). Second, we generate 10,000 simulations to estimate risk at two confidence levels (95% and 99%) and two time

Table 3 Parameters of simulated models

parameters	True value (θ)	Model	Description	Mean ($\hat{\theta}$)	std. dev.	t-prob ($\theta = \hat{\theta}$)
μ	5%	0	Drift	4.95%	0.0025	0.856
σ	20%	0	Volatility	19.70%	0.0038	0.433
α	5%	1	Reversion drift	6.12%	0.013	0.376
μ	10.50	1	Drift	10.506	0.026	0.834
σ	20%	1	Volatility	20.60%	0.020	0.765
μ_r	5%	2	Drift of drift	5.59%	0.014	0.668
σ_r	10%	2	Volatility of drift	8.25%	0.026	0.507
γ_r	15%	2	Reversion drift	15.23%	0.021	0.915
σ	20.0%	2	Volatility	19.75%	0.020	0.903
$\rho_{r,S}$	25%	2	Correlation return & drift	21.96%	0.041	0.466
μ	5%	3	Drift	3.99%	0.012	0.390
σ	20%	3	Volatility	22.19%	0.043	0.615
λ^+	2.50	3	Jump up	2.19	0.232	0.184
η^+	10%	3	Mean jump up	7.66%	0.018	0.194
δ^+	5%	3	Deviation jump up	4.03%	0.016	0.538
λ^-	2.00	3	Jump down	1.896	0.137	0.449
η^-	-8%	3	Mean jump down	− 6.23%	0.016	0.283
δ^-	3%	3	Deviation jump down	2.67%	0.012	0.792
μ	5%	4	Drift	3.96%	0.009	0.237
μ_σ	20%	4	Drift of volatility	19.86%	0.036	0.969
v	10%	4	Volatility of volatility	9.37%	0.009	0.488
γ_σ	2%	4	Reversion volatility	1.84%	0.005	0.738
$\rho_{S,\sigma}$	− 25%	4	Correlation return & volatility	− 20.29%	0.033	0.153
μ_r	5%	5	Drift of drift	5.33%	0.004	0.385
σ_r	10%	5	Volatility of drift	10.82%	0.019	0.663
γ_r	15%	5	Reversion drift	13.95%	0.053	0.843
μ_σ	20%	5	Drift of volatility	18.19%	0.030	0.545
v	10%	5	Volatility of volatility	11.23%	0.028	0.659
γ_σ	2%	5	Reversion volatility	2.23%	0.002	0.297
$\rho_{r,S}$	25%	5	Correlation return & drift	20.83%	0.072	0.563
$\rho_{S,\sigma}$	− 25%	5	Correlation return & volatility	− 21.32%	0.049	0.452
$\rho_{r,\sigma}$	− 15%	5	Correlation drift & volatility	− 14.40%	0.006	0.332
μ	5%	6	Drift	5.60%	0.016	0.703
μ_σ	20%	6	Drift of volatility	19.34%	0.013	0.625
v	10%	6	Volatility of volatility	11.27%	0.065	0.845
γ_σ	2%	6	Reversion volatility	2.36%	0.007	0.620
$\rho_{S,\sigma}$	− 25%	6	Correlation return & volatility	− 21.74%	0.020	0.113
λ^+	2.50	6	Jump up	2.25	0.501	0.613
η^+	10%	6	Mean jump up	11.47%	0.044	0.739
δ^+	5%	6	Deviation jump up	4.89%	0.002	0.556
λ^-	2.00	6	Jump down	1.95	0.162	0.757
η^-	− 8%	6	Mean jump down	− 8.22%	0.002	0.344
δ^-	3%	6	Deviation jump down	6.43%	0.025	0.182

Table 3 (continued)

Trueparameters are the actual values used in the simulation. *mean* is the mean of the estimated and transformed parameters, *std.dev*. is the standard deviation of the estimated parameters and *t-prob* is is t-value test

Table 4 Selection of models using adjusted R^2

Model	Bitcoin	Brent	Euro/Dollar	Eurstoxx-50	Gold
Model-0	81.96%	92.04%	96.77%	91.64%	74.28%
Model-I	87.24%	90.47%	97.69%	91.66%	74.39%
Model-II	89.14%	90.39%	90.78%	87.45%	84.92%
Model-III	84.81%	91.05%	96.53%	91.65%	74.33%
Model-IV	90.15%	89.28%	83.37%	94.22%	78.47%
Model-V	81.35%	86.35%	92.03%	83.26%	83.72%
Model-VI	93.65%	91.43%	92.32%	93.16%	89.51%

horizons (1 day and 5 days). For these simulations, we use the mean and percentile (95% and 99%) values of the parameters to calculate the market risk (*M*) and market plus model risks (*Mm*), respectively. For the selected model of each asset, Table 5 shows the mean and standard deviation of the parameters.

Figure 1, for example, shows daily market price, daily simulated prices at 95% confidence level from *M* and *Mm* methodologies, respectively. To make the visualization easier, we only present the last quarter of 2020 for the Eurostoxx-50 and Gold.

Tables 6 and 7 present risk estimates for time horizons of 1 day and 5 days, respectively.

For Tables 6 and 7, note for all sample confidence levels, time horizons and assets, the number of cases in which realized losses exceed those estimated is higher for the market risk estimate (*M*) than for the joint estimate of market and model risks (*Mm*). This means that the percentage of excesses is higher than expected (minus 1 confidence level) when estimating market risk, while if estimating market and model risks it is not exceeded in any case. The excess of realized loss beyond the estimated (excess loss) is also greater for *M* than for *Mm*. Finally, we verify that the cost of risk overestimate (cases in which the realized loss is less than the estimated loss), measured as an average of $\frac{oversetimate}{exceeded \quad loss}$, is also higher when we only estimate market risk.

In summary, including the model risk with the market risk provides the estimate procedure with a degree of flexibility setting the models and their parameters that allows meeting the requirements of the percentage of cases with excess losses (1 minus level of confidence) and at the same time presents lower excess losses and lower cost of overestimate over exceeded losses.

Table 5 Parameters of selected models

	Brent		Euro/Dollar		Eurstoxx-50		Bitcoin		Gold	
Model	Model-0		Model-I		Model-IV		Model-VI		Model-VI	
parameters	Mean	Std. dev.	Mean	Std. dev.	Mean	Std. dev.	Mean	Std. dev.	Mean	Std. dev.
μ	− 1.95%	8.271%	0.97%	0.49%	3.38%	1.46%	39.74%	4.30%	5.70%	2.13%
σ	39.35%	4.344%	12.67%	3.46%						
α			3.93%	1.41%						
μ_σ					21.98%	6.84%	82.39%	12.64%	29.04%	4.18%
γ_σ					3.69%	1.39%	10.38%	3.43%	7.28%	1.57%
ν					10.14%	4.40%	38.20%	5.95%	11.43%	3.43%
$\rho_{S.\sigma}$					− 6.74%	1.65%	− 9.85%	2.15%	− 5.65%	1.75%
λ^+							2.68	0.67	2.46	0.85
η^+							10.57%	4.02%	14.31%	1.28%
δ^+							5.02%	2.07%	6.90%	1.73%
λ^-							2.72	1.66	2.38	1.06
η^-							− 13.40%	2.86%	− 2.45%	4.02%
δ^-							3.20%	1.24%	4.08%	1.79%

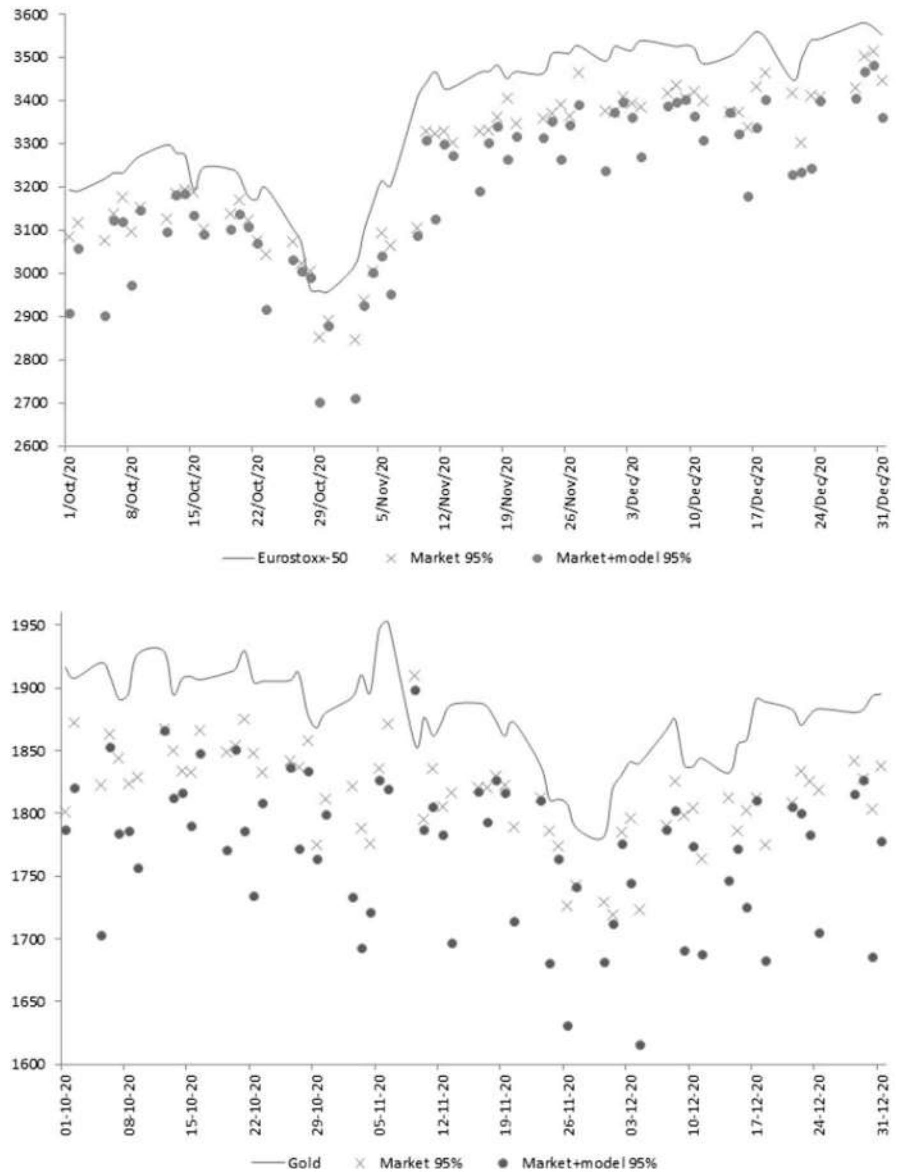

Fig. 1 Real prices and simulated prices from Market and Market+model risk methodologies

Conclusions

Since the last financial crisis in 2008, major legal changes have taken place in estimating risks by high-ranked financial institutions. This study focuses on the model risk implicit in the market risk estimates.

Table 6 Results of risk estimates for 1-day time horizon

Estimation	*M* at 95%	*M* at 99%	*Mm* at 95%	*Mm* at 99%
	Brent			
Number of losses exceeded	42	34	25	5
Average loss exceeded	1.17	1.14	1.06	1.03
Av. loss exceeded/av. overestimate	− 0.61	− 0.52	− 0.53	− 0.46
% losses exceeded	8.22%	6.65%	4.89%	0.98%
Average loss exceeded/mean price	2.17%	2.12%	1.97%	1.93%
	Bitcoin			
Number of losses exceeded	48	26	25	5
Average loss exceeded	315.18	304.86	306.19	294.00
Av. loss exceeded/av. overestimate	− 0.65	− 0.56	− 0.63	− 0.54
% losses exceeded	9.39%	5.09%	4.89%	0.98%
Average loss exceeded/mean price	3.39%	3.28%	3.30%	3.17%
	Euro/Dollar			
Number of losses exceeded	43	31	24	5
Average loss exceeded	0.07	0.05	0.05	0.04
Av. loss exceeded/av. overestimate	− 1.61	− 0.91	− 0.96	− 0.69
% losses exceeded	8.41%	6.07%	4.70%	0.98%
Average loss exceeded/mean price	0.59%	0.41%	0.45%	0.35%
	Eurostoxx-50			
Number of losses exceeded	18	13	14	3
Average loss exceeded	66.71	48.16	58.97	40.62
Av. loss exceeded/av. overestimate	− 0.59	− 0.38	− 0.51	− 0.31
% losses exceeded	3.52%	2.54%	2.74%	0.59%
Average loss exceeded/mean price	1.99%	1.44%	1.76%	1.21%
	Gold			
Number of losses exceeded	11	8	9	4
Average loss exceeded	15.88	10.87	10.40	8.80
Av. loss exceeded/av. overestimate	− 0.30	− 0.18	− 0.19	− 0.14
% losses exceeded	2.15%	1.57%	1.76%	0.78%
Average loss exceeded/mean price	0.98%	0.67%	0.64%	0.55%

M is market risk results, *Mm* is market and model risks joint estimation

There is no consensus in the financial literature on the best methodology to measure market and model risk jointly, although most studies point to the Monte Carlo simulation as the most suitable.

Following this alternative, this empirical study proposes a least squares methodology and a Monte Carlo simulation to select the stochastic model in discrete time that best adjusts to the behavior of asset market prices and at the same time estimates the corresponding parameters. Next, using the average values of these parameters, we calculate the market risk and using the percentiles (according to the desired

Table 7 Results of risk estimates for 5-days time horizon

Estimation	*M* at 95%	*M* at 99%	*Mm* at 95%	*Mm* at 99%
	Brent			
Number of losses exceeded	55	48	50	38
Average loss exceeded	4.12	3.23	3.41	3.05
Av. loss exceeded/av. overestimate	1.17	1.14	1.06	1.03
% losses exceeded	10.85%	9.47%	9.86%	7.50%
Average loss exceeded/mean price	7.67%	6.02%	6.35%	5.67%
	Bitcoin			
Number of losses exceeded	45	24	22	5
Average loss exceeded	566.12	545.94	543.96	530.47
Av. loss exceeded/av. overestimate	315.18	304.86	306.19	294.00
% losses exceeded	8.88%	4.73%	4.34%	0.99%
Average loss exceeded/mean price	6.10%	5.88%	5.86%	5.71%
	Euro/Dollar			
Number of losses exceeded	44	29	21	4
Average loss exceeded	0.06	0.05	0.04	0.03
Av. loss exceeded/av. overestimate	0.07	0.05	0.05	0.04
% losses exceeded	8.68%	5.72%	4.14%	0.79%
Average loss exceeded/mean price	0.50%	0.42%	0.36%	0.23%
	Eurostoxx-50			
Number of losses exceeded	16	12	13	3
Average loss exceeded	213.77	200.87	202.02	188.98
Av. loss exceeded/av. overestimate	66.71	48.16	58.97	40.62
% losses exceeded	3.16%	2.37%	2.56%	0.59%
Average loss exceeded/mean price	6.37%	5.99%	6.02%	5.63%
	Gold			
Number of losses exceeded	5	4	4	3
Average loss exceeded	49.74	38.11	39.62	27.23
Av. loss exceeded/av. overestimate	15.88	10.87	10.40	8.80
% losses exceeded	0.99%	0.79%	0.79%	0.59%
Average loss exceeded/mean price	3.08%	2.36%	2.46%	1.69%

M is market risk results and *Mm* is market and model risks joint estimation

level of confidence) of the estimated parameters we jointly determine the market and model risk.

First, we test this methodology on simulated data from previously selected models and parameters. The statistical results do not reject the null hypothesis about the equality of the real and the estimated parameters.

Later we apply the same methodology to market data. We select different kinds of assets (Bitcoin, Brent, Euro/Dollar exchange rate, Eurostoxx-50 and Gold) and a period between January 1, 2014 and the December 31, 2020. The results show that although we selected the stochastic model that best fits the data in discrete time, the estimate of market risk using the mean value of the parameters exceeds the maximum number of excess loss cases (1 minus confidence level) for all confidence levels and time horizons. However, the same is not true when we estimate the market risk and the joint model. The average excess of loss over the average price of the corresponding asset is always higher when we only estimate the market risk. Finally, we study whether the market risk methodology together with the model methodology leads to an overestimate of risk. We analyze the mean overestimate of the average exceeded losses and we verify that the cost of the overestimate of the proposed methodology is lower than that of estimating only the market risk.

This empirical work provides a methodology to jointly estimate market and model risks in a flexible, feasible and easy-to-implement manner. Our methodology presents consistent results that are superior to measuring market risk alone. The results of this empirical research are relevant for economic agents since the methodology presented allows them to use the same simulation procedure to calculate the risk of their portfolios and the risk of the model they assume in the valuation of their portfolios, thus complying with Basel regulatory requirements. In addition, this study also opens up the possibility of applying this methodology to other types of risk such as credit and operational.

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