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Where is the distribution tail threshold? A tale on tail and copulas in financial risk measurement☆

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ABSTRACT

Estimating the market risk is conditioned by the fat tail of the distribution of returns. But the tail index depends on the threshold of this distribution fat tail. We propose a methodology based on the decomposition of the series into positive outliers, Gaussian central part and negative outliers and uses the latter to estimate this cutoff point. Additionally, from this decomposition, we estimate extreme dependence correlation matrix which is used in the measurement of portfolio risk. For a sample consisting of six assets (Bitcoin, Gold, Brent, Standard&Poor-500, Nasdaq and Real Estate index), we find that our methodology presents better results, in terms of normality and volatility of the tail index, than the Kolmogorov–Smirnov distance, and its unnecessary capital consumption is lower. Also, in the measurement of the risk of a portfolio, the results of our proposal improve those of a t-Student copula and allow us to estimate the extreme dependence and the corresponding indexes avoiding the implicit restrictions of the elliptic and Archimedean copulas.

1. Introduction

In the field of financial markets, the study of the downside tail of the distribution of asset returns is a fundamental and essential factor for the analysis of market risks. This is a consequence of one of the stylized facts of asset returns, known as fat tail risk. Adequate risk estimate is fundamental in finance for different reasons such as asset pricing, portfolio management and financial risk measurement. The literature has shown that the tail of the distribution of asset returns does not follow a normal distribution (stylized facts), especially when the frequency of observation is high (for example, daily versus monthly) giving rise to a property of returns known as scaling.

The empirical studies (see Jansen et al., 2000) focus on studying the downward tail of the returns distribution by applying Generalized Extreme Value (GEV) distribution or Generalized Pareto Distribution (GPD), since show superior results for estimating risk compared to other parametric or non-parametric approaches. These distributions are defined by two parameters known as scale and shape which, as Brooks et al. (2005) showed, are related. Thus, knowing the shape parameter we can estimate the scale parameter. Therefore, in order to use extreme distributions, the shape parameter, also called the inverse of the tail index, must be estimated beforehand.

Hill (1975) is the usual estimator of tail index. However, as this method of estimation is not without its drawbacks. As Koedijk et al. (1990) pointed out, the problem lies in selecting an appropriate number of tail observations to include in the estimate of the tail index, since if we include too many data then the estimate variance is reduced but the estimate bias increases. However, if we consider too many data in the central range of distribution (or too few observations in tail) then the bias declines but the estimate variance is too large. So, the estimation of the index of the tail of the distribution presents the following difficulties: first, the observations should be *i.i.d.*; second, the low sample observations in the tail of the distribution to estimate the index; and third, the estimate is sensitive to the choice of the beginning (threshold) of the tail of the distribution. As a consequence of these difficulties, the literature has developed two approaches to solve the problems: modifying the method of calculating the tail index (for example Huisman et al., 2001) and seeking the best approximation to find the tail threshold of the distribution. Regarding the approach of using different tail index calculation methods, Fedotenkov (2020) reviews more than one hundred tail index estimators, discusses their assumptions and provides closed-form expressions and finds that some estimators perform better than others. Interestingly, the five estimators

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with the lowest error standard deviation are variants of the Hill estimator and have the same problem, the determination of the threshold of the tail of the distribution. The Hill estimator also performs well, with results among the top 10. In short, the problem is not the estimator (Hill) but determining the beginning threshold of the tail of the distribution. Then, the main drawback is to find the threshold of the tail distribution (the second approach studied in the literature).

In this context, some empirical studies arbitrarily set the tail threshold of the downward distribution at a percentile for example the 5th (Rhee & Wu, 2020). But, in addition to modifying Hill's estimator, the literature has also studied two methodologies for estimating the tail threshold of a distribution: bootstrapping method and fitting distribution GPD tail. Németh and Zempléni (2020) propose a double bootstrap method to estimate the Hill tail index but note that the computation time grows exponentially using either the Kolmogorov–Smirnov distance or the approximation of Hall (1990) to estimate the starting threshold of the tail of the distribution and the result convergence is also conditioned by sample size. Also, Drees et al. (2020) estimate the threshold value minimizing the Kolmogorov–Smirnov distance between the empirical distribution of the exceedances and the Pareto distribution adjusted to the larger order statistics, such that the threshold is the order statistic with the smallest Kolmogorov–Smirnov distance. From these empirical studies, we observe that Kolmogorov–Smirnov distance shows better results and lower computational time than bootstrap and double bootstrap methods.

In addition to the extreme univariate estimation of risk, the financial literature on the multivariate analysis of extreme movements in asset returns is vast. The application of extreme event theory to multivariate financial problems is a current topic that is being applied increasingly to more and more types of assets (see for example, Chan et al., 2022). Longin and Solnik (2001) highlight that using extreme value theory to model the multivariate distribution tails derives in the distribution of extreme correlation for a wide class of return distributions. Their results reject the null hypothesis of multivariate normality for the negative tail and they found that correlation increases in bear markets. However, they arbitrarily choose a logistic distribution (Gumbel copula) to adapt the extreme dependence. They note that tail index controls the level of dependence between extreme returns and this parameter is related with the correlation between extreme returns for pairs of assets. In addition to the inconvenience of selecting the starting threshold of the tail of the probability distribution, they use maximum likelihood procedure to estimate all parameters, so the computation time is very high. Kole et al. (2007) find that the Gaussian copula underestimates (too optimistic on diversification benefits) the probability of joint extreme downward movements, while the Gumbel copula overestimates (too pessimistic) this risk, so the Student's t copula is superior. To select copula, they use Kolmogorov–Smirnov and Anderson–Darling distances, but these measurements are sensitive to outliers. The procedure consists of two phases, estimating the parameters of the marginal distributions and then estimating the copula parameters, so this approach is very intensive computationally. To analyze the dependence between asset returns, Delatte and Lopez (2013) apply Gaussian, Clayton, and Gumbel copulas to consider three types of co-movement: (i) frequent and symmetrical, (ii) mostly present during extreme events and either symmetrical or asymmetrical, and (iii) mostly present during negative and extreme events, i.e. asymmetrical. The procedure consists of a two-stage maximum likelihood approach. First, they estimate and select the model that provides the best fit for the individual variables, and then they estimate the dependence structure of the copula. They find that the copula parameters are time-dependent and depending on the assets included in the portfolio then, the extreme dependence may present either symmetrical or asymmetrical behavior.

As a consequence of the difficulty of selecting a suitable copula for extreme dependencies by pairs of assets, Zhang et al. (2013) propose a tail dependence regression. This tail dependence index is modeled as a linear combination of the predictors through a monotonic transformation which is estimated by maximum likelihood. The results show that it captures more information about tail dependence.

In summary, while multivariate parametric risk estimation, such as the Gaussian, is computationally fast, its estimates underestimate extreme risk. On the other hand, using multivariate copulas improves risk estimate but is computationally slower (parameter estimation and simulation) and some copulas are complex to estimate when there are more than two assets (e.g. Gumbel). Additionally, Zhang (2008) and Embrechts et al. (2009) show that extreme dependence index not only measures the relationship between extreme movements of the downward tail of the distribution of two assets, but can also occur between the downward tail of one of the assets and the upward tail of the other. Thus, we note that in multivariate risk estimation there are several unresolved main issues: (i) how to approximate the dependence between assets, either using all observations (parametric) or only the extreme dependence (copula function); (ii) how to measure extreme dependence between asset returns; and (iii) the complexity and computation time of multivariate risk increases as the degree of risk adjustment improves.

The literature, after the last financial crises, has also focused on analyzing the extreme dependence relationships between asset returns. Harris et al. (2019) propose new systematic tail risk measures, but this proposal, as with the estimation of the tail index, has the disadvantage of fixing a priori the percentile (10th percentile) where the tail of the distribution begins or the threshold of the downward tail of the distribution. In addition, the literature has studied the dependence between assets and cryptocurrencies, as a consequence of the high volatility of the latter. Thus, Hussein Abdoh (2020) found tail dependence between returns for Bitcoin and other financial assets using the quantile cross-spectral dependence approach and, Ahelegbey et al. (2021) study extreme downside relationship among cryptocurrencies.

From the literature reviewed above, we note that in the individual risk estimate through applying the extreme event theory (VaR-GPD) there is a fundamental unresolved problem relating to the threshold of the tail of the distribution, necessary to estimate the tail index and keeping in mind that the data used in the estimate should be $i.i.d$. In addition, the analysis of extreme dependence is also conditioned by this threshold for the beginning of the distribution tail.

In this context, our aim is to develop a simple and computationally fast methodology to determine the threshold of the tail of the distribution. To do so, we follow González-Sánchez (2022) and divide each time series of returns into three independent series: Good the positive outliers, Usual the central part of the distribution with Gaussian behavior, and Bad the negative outliers. We use the methodology proposed by González-Sánchez (2021), who showed that some characteristics of asset returns (scaling, autocorrelation and heteroskedasticity) are caused by outliers.

Then, we compare the results of estimating risk using a VaR-GDP where the tail index is estimated using different methods and determining the downside tail starting threshold by minimizing the distance Kolmogorov–Smirnov with those obtained from the our methodological approach. Additionally, we analyze the extreme dependence between Bad–Bad data for each asset pair in a portfolio, since this proposal is simple and computationally faster than a copula.

Our proposal is tested on a sample of data composed of six assets traded in USD and representing assets with different characteristics (Bitcoin, Gold, Brent, Nasdaq index, Standard & Poors-500 index and Down Jones Real Estate index).

From the empirical results, the main contribution of this study is a new methodology to find the threshold of distribution tail which is faster computationally and better performance than Kolmogorov–Smirnov distance, in terms of tail index behavior (lower variance and more normality) and in terms of capital consumption due to the risk measurement. Additionally, our second contribution is a simple estimate of the extreme dependence between assets that is computationally less intensive than copulas, also shows multivariate risk overestimates lower than a t-Student copula and, unlike the extreme downside co-moment measures, the extreme dependence percentile is not a parameter whose value is arbitrarily fixed.

The rest of the paper is structured as follows: Section 2 develops the estimate methodology, Section 3 describes the sample, Section 4 shows and discusses the empirical results, and, finally, Section 5 presents the main conclusions.

2. Methodology

2.1. Methodology for univariate risk

Gençay and Selçuk (2004) noted that there is a simple relationship between the GPD and GEV distribution and they use GPD to estimate risk since they are not only interested in the maxima of observations but also in the behavior of large observations which exceed a high threshold and then, the Value at Risk using GPD is:

$$VaR_{\gamma} = r^{(m)} + \frac{\sigma}{\xi} \cdot [(\gamma \cdot \frac{T}{m})^{-\xi} - 1] \quad (1)$$

Where σ is the scale parameter, ξ is the shape parameter, γ is the confidence level at which the risk is estimated and, for a sample with size T, $r^{(m)}$ is the value of returns corresponding to the order statistic m ($r^{(1)} \leq r^{(2)} \leq \cdots r^{(m)} \leq \cdots \geq r^{(T)}$) or the threshold value where the tail of the distribution starts.

Since there are few observations in the tail of the distribution, the estimating the scale and shape parameters by maximum log-likelihood is complex, hence the method of moments, among others, is used. But, as Brooks et al. (2005) showed, there is a relationship between these two parameters, so that, knowing the shape parameter, the scale parameter is calculated as (see equation 13 in Brooks et al., 2005):

$$\sigma = (\frac{1}{m} \sum_{i=1}^{m} r_i^{\frac{1}{\xi}})^{\xi} \quad (2)$$

Then, we only need to estimate the shape parameter. The inverse of the shape parameter is known as the tail index of the distribution ($\alpha = \xi^{-1}$). Furthermore, when an economic agent estimates the risk of a portfolio using parametric methods and has to choose a probability distribution, then the tail index is a key factor, since this index is equal to the number of defined moments of the observed series of returns (or the degrees of freedom in the case of a t-Student distribution), and therefore, it would eliminate from the equation those distributions that would require more moments than the maximum determined by this index. This reflection has already been expressed by Koedijk et al. (1990) which pointed out the impossibility of testing the use of different non-nested probability distributions to adjust the behavior of asset returns and as a consequence, they proposed to focus on the analysis of the tail index.

Hill (1975) shows an approach to infer the tail behavior of a distribution without assuming any global form for the distribution function, but merely the tail's form of behavior. From the order statistics of a series ($r^{(1)} \leq r^{(2)} \leq \cdots r^{(m)} \leq \cdots \geq r^{(T)}$), the Hill estimator of shape parameter (ξ) is:

$$H_{simple,m} = \frac{1}{m} \sum_{i=1}^{m} ln \frac{r^{(i)}}{r^{(m)}} \quad (3)$$

From (3), note that, in practice, estimating the tail index is equivalent to determining the starting threshold of the tail of the distribution (m or $r^{(m)}$).

In this context, Jansen et al. (2000), using Value at Risk (VaR) for GPD, showed that extreme value theory proves to be a useful procedure for estimating VaR-efficient portfolios and for describing portfolio risk for events far out in the tails of the distribution. They estimate the tail index using the Hill estimator but as they point out (see footnote 8) the method for selecting the distribution tail threshold is not a universal remedy (function of the root of the sample size). Jondeau and Rockinger (2003) point out that estimators of tail index present two important drawbacks: first, the estimators are affected by the choice of the threshold (starting point of tail distribution) and second, tail index estimators are biased when the series is not i.i.d. As consequence, they use an approach does not consider the distribution of the tails but, rather, the distribution of the maximum or minimum returns over given subsamples. In this case, the limit distribution of extremes is still a generalized extreme value distribution, but an additional parameter, the extremal index, has to be included in the model and estimates by maximum likelihood. However, they replace the problem of determining the beginning of the tail of the distribution with the problem of correctly fixing the size of the subsamples, and the complexity of the estimate is increased by adding a new parameter and an optimization to estimate it. But, from Table-2 in Gençay and Selçuk (2004), we observe that the lower tail is different from the upper one, unlike what was found by Jondeau and Rockinger (2003), and we also note that the lower tail is different among assets. Brooks et al. (2005) compare different extreme value models for determining the value at risk, but once again the threshold is arbitrarily fixed at the 5th percentile. Rhee and Wu (2020) estimate a VaR GPD but as they note, following the common practice in the literature, for each estimate they choose the same percentile threshold for all assets in the sample (the top and bottom 20th percentiles as the up and down thresholds, respectively and with a minimum of 10 observations).

To avoid the drawback of tail threshold selection, the literature searches for other tail index estimation methods. So, Huisman et al. (2001) proposed a weighted least squared method to estimate the tail index:

$$H_{weighted} = \sum_{k=1}^{K} w_k \cdot H_k \quad (4)$$

Where H_k is the usual simple Hill estimator with starting downside tail in k order statistic, w_k is the weight calculated as $\sqrt{k}$ and K is the minimum order statistic considered, i.e., the maximum number of observations assumed in the downward tail of the return distribution. Thus, we obtain an order-weighted average Hill estimator. Also, Hill (2010) showed that the Hill estimator of the tail index is uniformly weakly consistent for processes such as ARFIMA and FIGARCH. This is because the data series is not $i.i.d$. Approaches such as the one used by McNeil and Frey (2000) present an additional complication and, as a consequence, Stupfler (2016) proposes an approach to estimate tail index in the presence of random covariates:

$$H_{double} = 1 - \frac{1}{2} \cdot (\frac{H^{(1)} \cdot H^{(1)}}{H^{(2)}})^{-1}, \quad (5)$$

Where $j = 1, 2$ then:

$$H^{(j)} = \frac{1}{m} \sum_{i=1}^{m} [ln \frac{(r^{(i)})}{ln(r^{(m)})}]^{j} \quad (6)$$

In this context, our proposal follows the Good–Usual–Bad (GUB) decomposition proposed by González-Sánchez (2021, 2022) to estimate the beginning of the threshold of the tail of the distribution. Thus, we define t as temporary moment of observation from a sample of size T, p is the asset log-price and r is asset return estimated for frequency 1-day as $r_t = p_t - p_{t-1}$. Then, we use the GUB methodology to divide the original series into three linearly independent sub-series to reflect positive outliers (Good or G), normal behavior (Usual or U) and negative outliers (Bad or B). To do this decomposition, first we select τ_n, τ_a, $\tau_{a,2}$ and τ_h as a test of normality, autocorrelation, autocorrelation for the square of the data and heteroskedasticity, respectively. For a sample of r_t, first we do $U_t = r_t$, run these tests on this sample U_t and if any of them show p-values lower than confidence level (for example, 0.05 or 0.1) then we search outliers as:

1. We seek $z_t = max|U_t|$.

2. According to the sign of the data, we do $G_t = z_t$ (positive) or $B_t = z_t$ (negative) and then, we substitute $U_t = 0$.
3. After replacement, we re-estimate τ_n, τ_a, $\tau_{a,2}$ and τ_h for the new series of U_t and, if we reject any hypotheses on normality, non-autocorrelation and non-heteroskedasticity, we go to back step (1). We only stop when all the null hypotheses of the tests applied on the U series are accepted (for more details on the GUB decomposition see González-Sánchez, 2021).

Once we have decomposed the original series, if the size of the Bad series is zero ($m_B = 0$), i.e., no negative outliers, the down tail index is 2 since distribution is Gaussian. But if $m_B > 0$ then we define the threshold of the downward tail of the distribution as $u_{m_B} = max(B_t)$ (or the highest value within the Bad sub-series).[1] Finally, we estimate the GUB tail index (α_{GUB}) as the inverse of Hill index:

$$H_{GUB} = \frac{1}{m_B} \sum_{i=1}^{m_B} ln \frac{B_i}{u_{m_B}} \qquad \alpha_{GUB} = H_{GUB}^{-1} \tag{7}$$

Then by applying expression-(2) we estimate the corresponding scale parameter and using expression-(1) we calculate the market risk. Note that u_{m_B} is not conditioned by the threshold arbitrarily set a priori, as usually in the literature, on the contrary, this threshold depends on the percentile of the starting point of the downward tail of the distribution of each asset returns.

2.2. Methodology for multivariate risk

The first multivariate estimate of risk for a portfolio (P) is the multivariate Gaussian and then, assuming equal weight for all assets results:

$$VaR_P^{\gamma} = \left[\begin{pmatrix} VaR_1^{\gamma} \\ \vdots \\ VaR_N^{\gamma} \end{pmatrix} \cdot \Omega \cdot \begin{pmatrix} VaR_1^{\gamma} & \cdots & VaR_N^{\gamma} \end{pmatrix} \right]^{0.5} \tag{8}$$

Where γ is the level of confidence at which the risk is estimated, VaR_i^{γ} is the estimated Gaussian univariate risk for the asset i that is part of the portfolio ($i = 1, \dots, N$) and Ω is correlation matrix among the portfolio's assets. Since this multivariate estimation assumes Gaussian behavior of both the individual assets and the portfolio, it has an advantage, fast computation, and a disadvantage, excess losses, since as we know from the stylized facts, both the assets and the portfolio do not usually have returns that fit normal behavior.

So, our proposal to estimate the multivariate risk of the set of assets ($i = 1, \dots, N$) that make up a portfolio with equal weighted,[2] first we calculate the GUB decomposition of asset returns time series and then, we estimate the univariate risk for each i-asset at confidence level γ as described above ($VaR_{i,GUB}^{\gamma}$). Next, we estimate the empirical extreme dependence among assets as the correlation matrix among the Bad series (Ω_B) obtained from the decomposition of the returns then, using the Cholesky decomposition, $\Omega_B = A_B \cdot A'_B$, so that the portfolio risk is as follows:

$$VaR_{P,GUB}^{\gamma} = \sum_{i=1}^{N} x_{i,GUB}^{\gamma} \tag{9}$$

Where $x_{i,GUB}$ is a component of the risk vector:

$$X_{GUB}^{\gamma} = \begin{pmatrix} A_B^{1,1} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_B^{1,i} & \cdots & A_B^{2,i} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_B^{1,N} & \cdots & A_B^{i,N} & \cdots & A_B^{N,N} \end{pmatrix} \cdot \begin{pmatrix} VaR_{1,GUB}^{\gamma} \\ \vdots \\ VaR_{i,GUB}^{\gamma} \\ \vdots \\ VaR_{N,GUB}^{\gamma} \end{pmatrix} \tag{10}$$

[1] Similarly, we define the threshold of the upwarp tail of the distribution as $u_{m_G} = min(G_t)$

[2] This approach can also be generalized to portfolios with different asset weights.

Note that this estimate is an empirical GUB multivariate risk estimation unlike parametric multivariate (e.g. Gaussian) and multivariate copulas (such as t-Student). So, this approach allows us to calculate a simple estimate of the extreme dependence or extreme correlation matrix among assets, but only using Bad component of original series of each asset returns. Besides, our proposal, unlike Harris et al. (2019), does not require setting the threshold of the tail of the distribution a priori, but rather the percentile is determined from the GUB decomposition of the original series of returns.

3. Data

The empirical study is applied on a sample composed of six assets traded in USD and selected for their representativeness of the stock market (Standard&Poors-500), technology companies (Nasdaq), energy market (Brent), safe-haven assets (Gold), the real estate market (Dow Jones Real Estate) and the cryptocurrency market (Bitcoin). The sample is composed of data at daily frequency from April 1, 2011 to May 31, 2022 obtained from Bloomberg. The sample period is justified by the inclusion of Bitcoin, since although it began trading in April 2010 it was not until a year later that it began to show any price volatility.

Daily returns are estimated from the daily prices as the daily difference of the log prices. A summary of the main statistics for daily returns and equal weighted portfolio is showed in Table 1.

From the results in Table 1, we observe that there are assets (Bitcoin) more volatile than others (Gold), that all of them are non-Gaussian and for only some of them (Gold and Brent) do not exist the autocorrelation problems. Also, the equal-weighted portfolio has the same statistical characteristics as the assets that compose it. In summary, the data seem to show the characteristics observed by González-Sánchez (2021) when studying the scaling property of asset returns and the effects of outliers. As such, the approach proposed in this empirical study is fully justified since the time series are not i.i.d., but the GUB decomposition avoids this problem in the analysis of the tails of the distribution.

4. Results

4.1. Empirical results for univariate estimates

Estimates are made at daily frequency for each of the six assets in the sample. We made a rolling estimate at each date using a database of the previous five years, as is usual in the financial field, so that at daily frequency the number of estimates for each asset is 1522 (2772 observations minus 1250 data from the five years prior to each calculation date).

First, we estimate the threshold of upside and downside tails distribution of the six assets. Table 2 shows the mean values and standard deviations (sd) of the 1522 rolling estimates for the six assets in the sample. We include the mean number of observations that remain in the tails or extreme values, mean return of threshold, mean percentile corresponding to threshold, tail index and mean Kolmogorov–Smirnov (KS) distance according to Drees et al. (2020). Also, we contrast the normality on the inverse of tail index, as it is well known (see among others Jansen et al., 2000):

$$(\hat{\alpha}_m^{-1} - \overline{\alpha}^{-1}) \cdot \hat{m}^{\frac{1}{2}} \sim \mathbf{N}(0, \overline{\alpha}_m^{-1}) \tag{11}$$

Where $\hat{\alpha}_m^{-1}$ is the inverse of tail index estimated at each date, $\overline{\alpha}^{-1}$ is the mean value of inverse the estimated indexes reported in Table 2, $\mathbf{N}$ is the cumulative normal distribution of zero mean and standard deviation $\overline{\alpha}^{-1}$ and, $\hat{m}$ is the order obtained in each estimation of the index as the threshold of the tail. For testing, we assume that mean value is the unknown true value.

From Table 2, notice that the KS distance obtained from GUB approach is very similar to minimum KS distance. The minimum average distances are smaller using the KS distance methodology for Nasdaq,

Table 1
Summary of statistics for daily returns.

Statistics	Bitcoin	Gold	Brent	Nasdaq	SP500	Real Estate	Portfolio
Observations	2772	2772	2772	2772	2772	2772	2772
min	−0.8488	−0.0982	−0.2798	−0.1315	−0.1277	−0.1916	−0.9076
mean	0.0038	0.0001	0.0002	0.0005	0.0004	0.0002	0.0050
std. Dev.	14.742	0.0578	0.2742	0.0893	0.0897	0.0916	0.0876
max.	0.0672	0.0103	0.0241	0.0128	0.0112	0.0127	1.4669
Skewness	28.182	−0.6204	−0.5770	−0.7297	−0.8515	−1.6298	0.3668
Excess Kurt.	29.7711	6.6897	22.3660	9.2408	15.7010	27.7900	40.886
Jarque–Bera	11064 [**]	5346.6 [**]	5793.1 [**]	1010.9 [**]	2880.9 [**]	9042.6 [**]	29314 [**]
LM ARCH(2)	21.542 [**]	26.486 [**]	50.407 [**]	60.002 [**]	77.526 [**]	50.674 [**]	53.331 [**]
Box–Pierce (2) raw	18.475 [**]	3.689	1.421	63.236 [**]	94.5941 [**]	64.079 [**]	29.587 [**]
Box–Pierce (2) squared	44.936 [**]	56.487 [**]	105.433 [**]	121.55 [**]	1463.70 [**]	94.3778 [**]	110.753 [**]

Note: $Jarque-Bera$ is a test on the normality of the series and whose null hypothesis is that the data are Gaussian. $LMARCH(lag)$ is a test of heteroscedasticity whose null hypothesis is the absence of heteroscedasticity. $Box-Pierce(lag)$ raw is a test of the autoregressivity of the series whose null hypothesis is the absence of autoregressivity. $Box-Pierce(lag) squared$ is a test on the autoregressivity of the square of the series whose null hypothesis is the absence of autoregressivity. [$*$] and [$**$] show that the null hypothesis of the test is rejected at 5% or 1%, respectively.

Table 2
Threshold results.

Threshold method	Estimates	Bitcoin	Gold	Brent	Nasdaq	SP500	Real Estate
Upside tail							
	mean observations	232	200	73	77	82	64
	mean threshold percentile	0.1740	0.1430	0.0570	0.0470	0.0510	0.0390
	mean threshold value	0.0344	0.0083	0.0305	0.0169	0.0138	0.0163
	sd threshold value	0.0031	0.0009	0.0012	0.0015	0.0010	0.0019
KS distance	mean tail index	1.6485	2.8435	2.3326	2.8041	2.5391	2.5517
	sd tail index	0.3617	0.5413	0.3068	0.8422	0.5730	0.6053
	mean KS	0.0861	0.0257	0.0549	0.0221	0.0124	0.0251
	Jarque–Bera	10.0066 [0.007]**	5.8388 [0.054]	2.113 [0.348]	6.4066 [0.041]*	6.5581 [0.038]*	9.849 [0.007]**
	mean computing time	3 min 6 s	2 min 52 s	2 min 55 s	2 min 49 s	2 min 54 s	3 min 2 s
	mean observations	219	71	69	135	160	128
	mean threshold percentile	0.1629	0.0605	0.0622	0.1052	0.1155	0.0929
	mean threshold value	0.0374	0.0150	0.0302	0.0130	0.0099	0.0117
	sd threshold value	0.0093	0.0033	0.0048	0.0040	0.0023	0.0026
GUB	mean tail index	1.5659	2.0550	2.3602	2.9377	2.7010	2.6508
	sd tail index	0.1839	0.0669	0.2624	0.7407	0.6366	0.7550
	mean KS	0.0857	0.0255	0.0547	0.0260	0.0134	0.0266
	Jarque–Bera	7.598 [0.022]*	5.4345 [0.066]	1.3174 [0.518]	4.9234 [0.085]	5.7016 [0.058]	5.3984 [0.067]
	mean computing time	1 min 55 s	1 min 46 s	2 min 3 s	2 min 11 s	1 min 52 s	2 min 8 s
Downside tail							
	mean observations	192	97	91	128	131	139
	mean threshold percentile	0.8580	0.9340	0.9210	0.9150	0.9130	0.9030
	mean threshold value	−0.0315	−0.0131	−0.0289	−0.0131	−0.0102	−0.0114
	sd threshold value	0.0031	0.0015	0.0021	0.0016	0.0010	0.0011
KS distance	mean tail index	1.4916	3.2625	2.6110	2.1259	1.8755	2.1141
	sd tail index	0.331	0.853	0.458	0.695	0.554	0.566
	mean KS	0.089	0.0483	0.0139	0.0625	0.0646	0.0914
	Jarque–Bera	6.3796 [0.041]*	3.015 [0.221]	3.5313 [0.171]	4.8281 [0.089]	2.5218 [0.283]	6.346 [0.042]*
	mean computing time	3 min 8 s	2 min 57 s	3 min 4 s	2 min 38 s	2 min 46 s	2 min 59 s
	mean observations	166	80	88	130	133	136
	mean threshold percentile	0.8759	0.9376	0.9239	0.9066	0.9052	0.8995
	mean threshold value	−0.0374	−0.0149	−0.0300	−0.0130	−0.0099	−0.0117
	sd threshold value	0.0093	0.0032	0.0046	0.0040	0.0023	0.0025
GUB	mean tail index	1.3452	2.7091	2.4949	2.1382	1.8912	2.1241
	sd tail index	0.1584	0.2828	0.2478	0.2170	0.3555	0.1159
	mean KS	0.086	0.0474	0.0132	0.064	0.0661	0.0917
	Jarque–Bera	6.11 [0.047]*	3.9687 [0.137]	1.8415 [0.398]	4.3855 [0.112]	2.129 [0.345]	5.9305 [0.052]
	mean computing time	2 min 4 s	1 min 48 s	1 min 53 s	2 min 6 s	1 min 57 s	1 min 54 s

Note: $KS\ distance$ is Kolmogorov–Smirnov minimum distance method, GUB is Good–Usual–Bad method, $mean\ observations$ is the mean number of observations in the tail of the distribution, $mean\ threshold\ percentile$ is the mean starting percentile of the distribution tail, $mean\ threshold\ value$ is the value corresponding to the mean starting percentile of the tail, $sd\ threshold\ value$ is the standard deviation of all estimated tail starting values, $mean\ tail\ index$ is the mean tail index estimated as a Hill indicator, $sd\ tail\ index$ is the standard deviation of the estimated Hill indexes, $mean\ KS$ is the mean value of the Kolmogorov–Smirnov distance between the tail of the distribution and a GPD distribution, $Jarque-Bera$ is the test of normality on inverse of tail index estimates, $mean\ computing\ time$ is the mean time of the estimates.

SP500 and Real Estate, but larger than those resulting from using the GUB methodology for Bitcoin, Gold and Brent. Now, when we contrast the normality of the inverse of the tail index, we find that both methodologies reject the hypothesis for Bitcoin but, in addition, the KS distance does so for Nasdaq, SP500 and Real Estate in the upside tail and for Real Estate in the downside tail.

As for the tail index, note that the average value is above 2 in both tails and for all assets except for Bitcoin and for the SP500 down tail. Now, if we consider $(mean - 2 \cdot sd)$, then all assets could have an index below 2.

We also note that in both the upside and downside tails for Bitcoin, Gold and Brent, the KS distance methodology considers a higher

Table 3
Analysis of central part of distribution.

Threshold method	Estimates	Bitcoin	Gold	Brent	Nasdaq	SP500	Real Estate
	mean observations	826	953	1090	1045	1037	1047
	Jarque–Bera	42.2391 [0.000]**	112.97 [0.000]**	0.9921 [0.609]	12.621 [0.001]**	47.565 [0.000]**	26.177 [0.000]**
KS distance	LM ARCH(2)	6.9457 [0.001]**	0.3876 [0.678]	1.9173 [0.1472]	0.0712 [0.931]	0.5993 [0.549]	1.5634 [0.209]
	Box–Pierce (2) raw	11.1351 [0.004]**	3.855 [0.145]	7.3927 [0.025]*	3.4901 [0.174]	2.5393 [0.281]	0.6576 [0.719]
	Box–Pierce (2) squared	14.0646 [0.001]**	0.7829 [0.676]	3.9143 [0.141]	0.1463 [0.929]	1.2172 [0.544]	3.2179 [0.201]
	mean observations	865	1099	1093	985	957	986
	Jarque–Bera	3.0468 [0,137]	0.6036 [0.739]	1.7324 [0.421]	3.3450 [0.188]	2.1536 [0.341]	4.8531 [0.089]
GUB	LM ARCH(2)	2.9172 [0.098]	0.5691 [0.566]	1.4447 [0.236]	2.2719 [0.103]	1.5026 [0.223]	0.3592 [0.698]
	Box–Pierce (2) raw	2.9598 [0.206]	2.1501 [0.341]	4.1965 [0.115]	4.6971 [0.129]	2.5305 [0.282]	0.1291 [0.937]
	Box–Pierce (2) squared	3.8961 [0.158]	1.1378 [0.566]	2.9182 [0.232]	4.7019 [0.095]	2.9597 [0.228]	0.7433 [0.689]

Note: $Jarque--Bera$ is a test on the normality of the series and whose null hypothesis is that the data are Gaussian. $LMARCH(lag)$ is a test of heteroscedasticity whose null hypothesis is the absence of heteroscedasticity. $Box--Pierce(lag)$ raw is a test of the autoregressivity of the series whose null hypothesis is the absence of autoregressivity. $Box-Pierce(lag)squared$ is a test on the autoregressivity of the square of the series whose null hypothesis is the absence of autoregressivity. [$*$] and [$**$] show that the null hypothesis of the test is rejected at 5% or 1%, respectively.

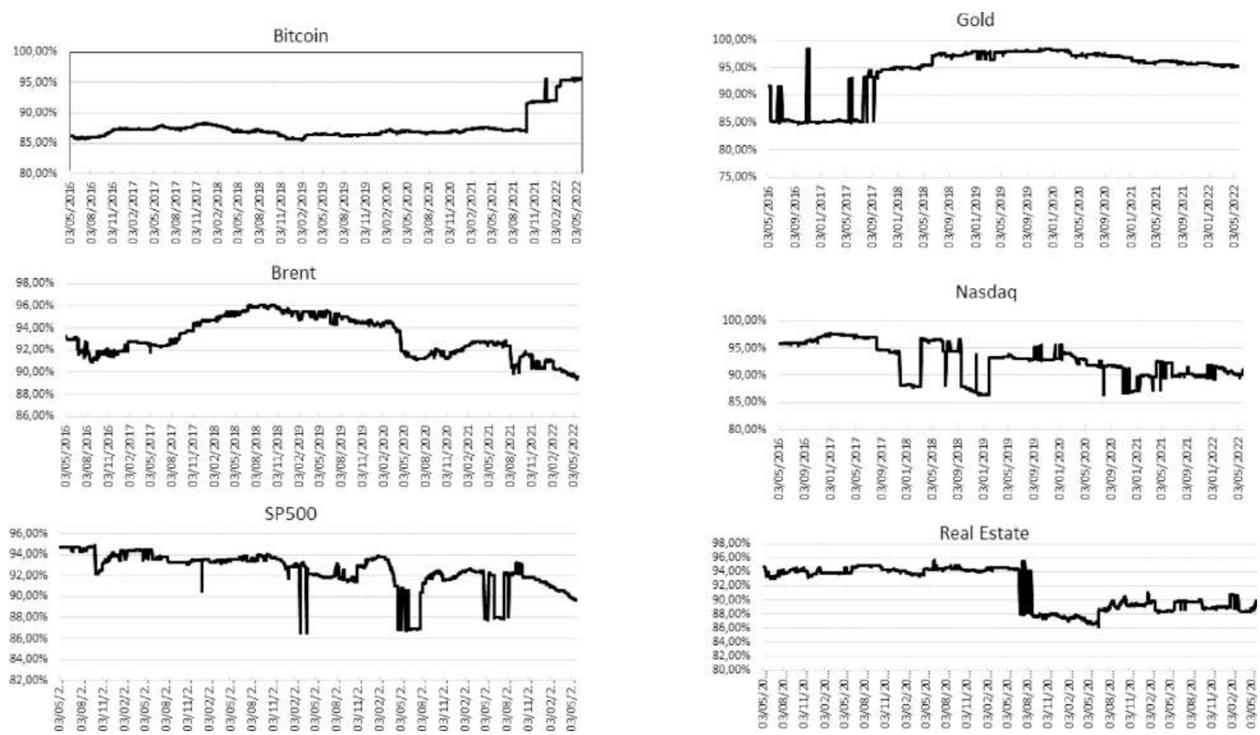

Fig. 1. Thresholds of the downward tail of the distribution of daily returns.

number of observations in the tails, while it includes a lower number than the GUB methodology in the cases of Nasdaq, SP500 and Real Estate. This difference in the volume of data included in the tails in each methodology means that the higher the number of observations in the tail, the higher the estimated index, logically since the distances between the threshold and the values of the last orders of the series is greater. Therefore, as Huisman et al. (2001) noted the number of observations included in the estimation conditions the result and the variability of the result. Regarding the latter, note that the standard deviation of the estimated tail index is lower in all cases using the GUB methodology than using the KS distance, probably because while the KS distance seeks to fit as well as possible the observations of the tail to a GDP, but with the drawback of using a low number of observations, the GUB methodology on the contrary, fits the central part of the distribution to a normal distribution i.i.d., logically using a larger volume of data, since it analyzes the whole series.

In short, the number of observations included in the tail conditions the value of the index, its volatility, the KS distance and the Gaussian behavior of the inverse of the tail index. To check this conclusion in Table 3 we show the average statistical tests of the central part of the returns series, i.e. the returns lower than the threshold of the up-tail and higher than the threshold of the down-tail.

From Table 3, note that the central part of the distributions fits quite well to a normal i.i.d., as indicated by González-Sánchez (2021), this is due to the exclusion of the outliers that make the statistical tests reject all hypotheses (compare with the results in Table 1). Now, the KS distance methodology fails to make the central part of the Bitcoin and Brent distribution i.i.d., as they reject the hypotheses of absence of autocorrelation (Brent and Bitcoin) and heteroscedasticity (Bitcoin). In addition, the KS distance, unlike the GUB methodology, does not accept the hypothesis of normality of the central part of the distribution and therefore, this indicates that either extreme values have been included in the central part or some others that do fit the Gaussian behavior have been excluded.

In summary, the GUB methodology performs better than the KS distance methodology for estimating the tail threshold of a distribution because it considers the entire data series, instead of using only the extreme values. In addition, it is computationally faster.

Fig. 1 shows the evolution of thresholds of down tail for each asset obtained by GUB methodology for daily frequency and thus check how it adjusts to the behavior of the returns series.

Table 4
Tail index results.

Method	Statistics	Bitcoin	Gold	Brent	Nasdaq	SP500	Real Estate
Upside tail							
Simple	mean index tail	1.5659	2.0550	2.3602	2.9377	2.7010	2.6508
	sd index tail	0.1839	0.0669	0.2624	0.7407	0.6366	0.7550
	min index tail	1.1815	1.9630	1.8551	1.8335	1.8026	1.8581
	max index tail	2.0744	2.2265	3.0293	4.9766	4.5898	4.0456
Double	mean index tail	1.5276	2.2195	2.5111	3.0940	2.9544	2.6715
	sd index tail	0.2195	0.0748	0.2902	0.9186	0.8153	0.8167
	min index tail	1.0966	1.9057	1.5448	1.3498	1.7394	1.8291
	max index tail	2.5853	2.4685	3.4774	5.1531	3.9669	3.8802
Weighted	mean index tail	1.2617	2.4012	2.5647	3.0098	2.6861	2.6724
	sd index tail	0.2339	0.0877	0.3172	0.8961	0.7733	0.7976
	min index tail	1.0717	1.9390	1.4414	1.3019	1.8981	1.8462
	max index tail	2.8951	2.5189	3.5196	5.0932	4.1925	3.9605
Downside tail							
Simple	mean index tail	1.3452	2.7091	2.4949	2.1382	1.8912	2.1241
	sd index tail	0.1584	0.2828	0.2478	0.2170	0.3555	0.1159
	min index tail	1.1694	1.8178	1.7864	1.5188	1.3136	1.8141
	max index tail	1.8167	3.4631	3.0311	2.7327	2.8447	2.4548
Double	mean index tail	1.3016	2.8478	2.6405	2.2808	2.1253	2.1255
	sd index tail	0.1938	0.2886	0.2742	0.3945	0.5326	0.1769
	min index tail	1.0835	1.7517	1.4629	1.0310	1.2441	1.7708
	max index tail	2.3229	3.7014	3.4548	2.9033	2.2041	2.2757
Weighted	mean index tail	1.0347	3.0247	2.6940	2.2014	1.8613	2.1371
	sd index tail	0.2065	0.3012	0.2995	0.3705	0.4877	0.1580
	min index tail	1.0586	1.7902	1.3631	0.9823	1.4048	1.7878
	max index tail	2.6215	3.7405	3.5003	2.8265	2.4327	2.3650

Note: $Simple$ is estimated using expression-(3), $Double$ by expression-(5) and $Weighted$ according to expression-(4). For rolling estimates: $mean$ is the average value, sd is standard deviation, min is minimum value and max is maximum value.

In Fig. 1, the closer to 100%, the more Gaussian the distribution will be, therefore, the fewer the outliers. For example, in the first estimates the threshold for Gold is at the 86th percentile, and from October 2017 onwards it is around 97th. In contrast, Bitcoin is around the 86th percentile until October 2021, at which time it begins to rise to the 96th percentile.

Once we have contrasted that the GUB methodology shows better results for estimating the tail threshold of the distribution, we analyze whether using different methods for calculating the tail index with GUB threshold, the variability of the estimates is lower than the simple Hill index. Table 4 shows the results:

From Table 4 we observe that Hill-$simple$ is the least volatile indicator (lower standard deviation or sd) and shows the lowest amplitude (max–min) for all assets. In short, if the threshold is well-fitted, it is not necessary to apply modifications to the Hill estimator to obtain a tail index with low variability.

Finally, we test the effectiveness of the GUB methodology (for estimating the threshold of the downside tail of returns) from the degree of fit of the univariate risk estimation using the expression-(1) and previously calculating scale parameter by expression-(2). We compare the results with KS distance. Table 5 shows the results:

Note, in the results in Table 5, that both methods of estimating the tail threshold obtain a percentage of exceeded losses lower than exceeds the maximum limit (1 minus the confidence level both at 95% and 99%), except KS distance method for Bitcoin and at 99% confidence level. Bitcoin is the asset with highest percentage of exceeded losses: for GUB method is 2.081% and 0.874% at 95% and 99% confidence level, respectively; while for KS distance is 3.043% and 1.237%.

For GUB method, we observe that the average excess of realized losses per $100 invested is between $-9.92 (Bitcoin) and $-2.35 (Gold) at 95% confidence level and, between $-5.86 (Bitcoin) and $-1.22 (Gold) at 99%; while for KS distance method is between $-12.35 (Bitcoin) and $-2.79 (Gold) at 95% and $-7.09 (Bitcoin) and $-1.48 (Gold) at 99%.

Regarding the average overestimate of risk, note that, for GUB method, the values are between $7.28 (Bitcoin) and $1.96 (SP500) at 95%, while at 99% confidence level the range is $3.17 (Bitcoin) and $0.71 (SP500). For KS distance method, the range are $9.21 (Bitcoin) and $2.14 (SP500) at 95%, while at 99% the values are between $4.52 (Bitcoin) and $0.88 (SP500).

In short, when we use the GUB methodology to estimate the tail index then, we obtain a VaR GPD with a percentage of exceeded losses does not exceed for any asset and confidence level the limit of (1 — confidence level) and lower percentage of exceeded losses than KS distance method. Also, both overestimation and underestimation of risk over the actual outcome for GUB method is lower than using KS distance method. Finally, we compare the ratio of capital consumption due to risk overestimation over risk underestimation for both methods and we find that, for all assets, if we use the KS distance method, the capital consumption due to risk overestimation for $1 of risk underestimation is higher than using the GUB method. Therefore, the results of the GUB method in the estimation of univariate risk by VaR GPD are better than those obtained by the KS distance method.

4.2. Empirical results for multivariate estimates

To analyze the multivariate GUB results in risk measurement, we first show in Table 6 the mean correlations among asset returns. We estimate the correlation as Kendall rank correlation since (Embrechts et al., 2009), among others, pointed out that this estimation method produces less volatile (more robust) results with respect to outliers than other methods such as Pearson's correlation.

The results in Table 6 shows that in daily frequency estimating the correlation matrix over the total sample either using Kendall's coefficient underestimates the extreme dependence between assets. For example, in the case of Gold, if we use the whole sample, the correlations are negative with the majority of assets, which would indicate that on average it is a safe haven asset, which is useful to know for pricing assets; but if we estimate the correlation using only the negative outliers of the series (Bad data), we find that the correlation is positive in all cases, so that Gold behaves like the other assets in the sample when faced with extreme events.

Table 5
Value at Risk estimate per $100 invested in each asset.

Method	Level	Estimate	Bitcoin	Gold	Brent	Nasdaq	SP500	Real Estate
KS distance	scale	mean value	1.1142	0.9571	0.9823	0.9138	0.9017	0.9109
GUB		mean	1.0131	0.9252	0.9557	0.9346	0.9221	0.9245
KS distance	95%	% excess	3.043%	1.357%	1.602%	1.596%	1.538%	1.872%
		mean exc.	−12.35	−2.79	−3.98	−3.91	−3.94	−5.38
		mean over.	9.21	2.61	2.37	2.66	2.14	4.31
		$\left\|\frac{meanover.}{meanexc.}\right\|$	0.75	0.94	0.60	0.68	0.54	0.80
	99%	% excess	1.237%	0.622%	0.611%	0.495%	0.513%	0.894%
		mean exc.	−7.09	−1.48	−1.95	−1.93	−1.97	−2.94
		mean over.	4.52	1.36	1.01	1.23	0.88	2.29
		$\left\|\frac{meanover.}{meanexc.}\right\|$	0.64	0.92	0.52	0.64	0.45	0.78
GUB	95%	% excess	2.081%	1.231%	1.613%	1.657%	1.526%	1.789%
		mean exc.	−9.92	−2.35	−4.19	−4.07	−3.84	−5.21
		mean over.	7.28	2.06	2.44	2.72	1.96	3.99
		$\left\|\frac{meanover.}{meanexc.}\right\|$	0.73	0.88	0.58	0.67	0.51	0.77
	99%	% excess	0.874%	0.412%	0.643%	0.557%	0.486%	0.766%
		mean exc.	−5.86	−1.22	−1.99	−2.03	−1.82	−2.79
		mean over.	3.17	1.05	1.02	1.28	0.71	2.11
		$\left\|\frac{meanover.}{meanexc.}\right\|$	0.54	0.86	0.51	0.63	0.39	0.76

Note: *%excess* is the ratio of the number of dates on which the outcome was worse than the estimated risk to the total number of dates; *mean exc.* is the average excess of actual losses over estimated risk when such losses were worse than the risk; *mean over.* is the average excess of the risk over the actual result when such risk was lower than the result obtained (overestimate).

Table 6
Mean Kendall correlation matrix.

Assets	Bitcoin	Gold	Brent	Nasdaq	SP500	Real Estate
			For all data			
Bitcoin	1	0.0308	0.0206	0.0512	0.0503	0.0389
Gold	0.0308	1	0.0441	−0.0292	−0.0314	
Brent	0.0206	0.0441	1	0.1694	0.2058	0.1148
Nasdaq	0.0512	−0.0292	0.1694	1	0.7911	0.4230
SP500	0.0503	−0.0314	0.2058	0.7911	1	0.4996
Real Estate	0.0389	0.0427	0.1148	0.4230	0.4996	1
			For Bad data			
Bitcoin	1	0.0459	0.0676	0.1191	0.1331	0.1104
Gold	0.0459	1	0.0758	0.0640	0.0831	0.1482
Brent	0.0676	0.0758	1	0.2501	0.3058	0.2327
Nasdaq	0.1191	0.0640	0.2501	1	0.8764	0.5092
SP500	0.1331	0.0831	0.3058	0.8764	1	0.5971
Real Estate	0.1104	0.1482	0.2327	0.5092	0.5971	1

Now, based on the individual asset risk estimates and using the correlation matrices estimated in each moment of sample period, we calculate the risk of the equal-weighted portfolio for a time horizon of 1 day and confidence levels of 95% and 99%. To do so, we apply the GUB methodology described above (see expressions (9) and (10)) and for comparison, we also estimate multivariate Gaussian risk (see expression (8)) and t-Student copula with 10,000 simulations and fitting the freedom degree by maximum log-likelihood. The results are shown in Table 7.

The results in Table 7 show that only the Gaussian multivariate estimate exceeds the allowed percentage of loss (1 minus confidence level). Also, we note that t-Student copula shows lower number of excess on losses than GUB method, as a consequence, we observe that the mean excess of realized loss over estimated loss for the t-Student copula is lower than for the GUB methodology, but the overestimation of risk over actual return (when the actual return is no worse than the estimated loss) for the t-Student copula is higher than for the GUB methodology. Therefore, GUB method consumes less capital unnecessarily (overestimation) per $1 of loss in excess of the estimated loss than the t-Student copula, in particular, while GUB approach show values of $1.16 and $1.46 at 95% and 99%, respectively; t-Student copula presents values of $1.98 (95%) and $3.07 (99%). In short, the risk overestimation of the t-Student copula versus the GUB method is much higher than the excess loss of the latter versus the t-Student copula, so that the higher unnecessary capital consumption of the copula makes its performance lower than the GUB approach. In addition, the GUB approach is computationally faster than the t-Student copula.

Finally, to analyze the versatility of the GUB methodology, we estimate the extreme dependence indexes between the assets that make up the portfolio. From Zhang (2008) and Embrechts et al. (2009), we know the total tail dependence for a pair of assets is the extreme dependence between up–up, up–down, down–up and down–down tails (for more details see expressions 5.1 to 5.4 in Embrechts et al., 2009). Besides, these empirical research point out that Archimedean copulas have no negative tail dependence and, elliptical distribution has symmetrical extreme dependence, i.e., up–down dependence is equal to down–up for each pair of assets. Thus, compared to this methodology, the GUB decomposition is more flexible since it allows determining the extreme dependencies (correlations) and also the extreme dependence indexes empirically. GUB approach is more ease and does not assume the symmetry of the elliptic distributions nor the non-negativity of the extreme dependence of the Archimedean copulas. Then, extreme dependence indexes (λ) between two pair of assets i and j are:

$$\lambda_{i,j}^{z_i,z_j} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{1}(z_{i,t}\cdot z_{j,t}\neq 0) \qquad (12)$$

Where z is Good, Usual or Bad values for i-asset and j-asset and then, we obtain nine dependence index, i.e., extreme dependence indexes (Good–Good, Good–Bad, Bad–Good and Bad–Bad) and, we also find the normal dependence (Usual–Usual, Usual–Good, Usual–Bad, Good–Usual and Bad–Usual).

Table 8 shows the Kendall rank correlation for all combinations (Good–Usual–Bad) and for each pair of assets.

From results in Table 8, we note that when the decomposition component of the series is the same (e.g. Good–Good), the correlation matrix is symmetric, whereas when the decomposition component is different (e.g. Good–Usual) the correlation matrix is asymmetric. For example, the Good–Good correlation between SP500 and Real Estate is 0.62, whereas the Good (SP500)-Bad(Real Estate) correlation is 0.0825, while the Good (Real Estate)-Bad(SP500) correlation is 0.0778. Therefore, GUB methodology is flexible enough to measure the dependence between assets and respecting the lack of asymmetry between the different parts of the probability distributions.

Analyzing the results of Table 8, we observe that when both assets show the same behavior (Good–Good, Usual–Usual and Bad–Bad) the significant correlations occur between Brent, Nasdaq, SP500 and Real Estate. In the case of Good–Usual and Usual–Good, Brent is the only

Table 7
Results for multivariate estimate of risk per $100 invested in equal-weighted portfolio.

Method	Confidence Level	Estimate	Value
Gaussian multivariate	95%	% excess	5.945%
		mean excess	−13.32
		mean overestimate	13.45
		$\lvert \frac{meanover.}{meanexc.} \rvert$	1.01
		mean computing time	2 min. 18 s
	99%	% excess	1.315%
		mean excess	−9.82
		mean overestimate	15.61
		$\lvert \frac{meanover.}{meanexc.} \rvert$	1.59
		mean computing time	2 min. 21 s
t-Student copula	95%	% excess	1.022%
		mean excess	−8.17
		mean overestimate	16.22
		$\lvert \frac{meanover.}{meanexc.} \rvert$	1.98
		mean computing time	18 min. 52 s
	99%	% excess	0.938%
		mean excess	−6.25
		mean overestimate	19.21
		$\lvert \frac{meanover.}{meanexc.} \rvert$	3.07
		mean computing time	18 min. 56 s
GUB decomposition	95%	% excess	1.735%
		mean excess	−8.81
		mean overestimate	10.23
		$\lvert \frac{meanover.}{meanexc.} \rvert$	1.16
		mean computing time	12 min. 33 s
	99%	% excess	0.980%
		mean excess	−7.64
		mean overestimate	11.17
		$\lvert \frac{meanover.}{meanexc.} \rvert$	1.46
		mean computing time	12 min. 41 s

Note: $\%excess$ is the ratio of the number of dates on which the outcome was worse than the estimated risk to the total number of dates; $mean\ exc.$ is the average excess of actual losses over estimated risk when such losses were worse than the risk; $mean\ over.$ is the average excess of the risk over the actual result when such risk was lower than the result obtained (overestimate), $mean\ computing\ time$ is the mean time of the estimates.

asset that shows a relationship with the rest, except for Gold and Real Estate. When the relationship is Usual–Bad and Bad–Usual, Gold is the only asset that shows no relationship in any sense with the rest. Finally, in the Good–Usual and Usual–Good relationships, Gold and Real Estate are the ones that show this type of dependence with respect to the rest of the assets except Bitcoin and Brent. In summary, Bitcoin does not show a significant relationship with any of the other assets except in the Usual situation and with Brent. Brent shows relationship with all assets except Gold for Good and Usual situation. The highest correlations are observed between SP500 and Nasdaq. Finally, Gold, and to a lesser extent Real Estate, are the only assets that show a significant Good–Bad relationship.

Finally, Table 9 shows the dependency indexes for each pair of assets using expression-(12).

From the results of the Table 9, note that, when the behavior of the assets is the same (Good–Good, Usual–Usual and Bad–Bad), Gold is the only asset that does not show a significant dependence index with the rest of the assets. When the relationship is one of the following: Good–Usual (Usual–Good) and Bad–Usual (Usual–Bad) then, we find that Bitcoin shows the most significant dependence indexes, i.e., in a Usual situation, investors seek a more extreme position (Good or Bad) that can bring them higher profits by assuming greater risks. Finally, in the Good–Bad (Bad–Good) situation, we again find that, as with the correlations, Gold is the asset with the highest number of significant extreme dependence indexes, this mean that the investor perceives it as a safe-haven asset in situations of extreme risk.

5. Conclusions

The estimate of the market risk of assets is a field of research within finance that is constantly evolving in response to the needs of economic agents in asset management and compliance with risk regulations.

Unlike asset pricing, which analyzes the entire distribution of returns, risk analysis focuses on the downside tail of returns. Furthermore, this analysis involves two parameters that are usually set a priori: the time horizon and the confidence level. While the former must take into account the economic agent's investment term policy and the liquidity of the asset, the latter does not seem to depend on any objective criterion. Moreover, in common practice, both parameters are usually set arbitrarily in financial regulation.

The objective of this empirical study is to help economic agents and regulators to objectively determine the level of confidence in risk estimates. To do so, we analyze the index of the downside tail of the return's distribution.

There is a vast literature on estimating the tail index, but there is no consensus on how to determine the threshold of the downside tail of the returns, i.e., where the tail of the distribution begins and therefore with what data we have to carry out the estimate of the index. This is highly relevant, since the literature has shown that risk estimates using the generalized Pareto distribution to adjust the tail of the distribution obtain more consistent results than other proposals that adjust the total distribution. However, using this distribution requires an adequate estimate of the beginning of the tail, since this conditions the subsequent estimate of the tail index, and this, in turn, conditions the estimate of the scale parameter.

We propose a method to determine the beginning of the downward tail of the distribution that is based on González-Sánchez (2021) and that allows us to separate the return series of an asset into three linearly independent components (GUB) so that the tails would be reflected by the positive and negative outliers, respectively, while the central part of the distribution is adjusted to a normal distribution.

For a sample of six USD-quoted assets representative of different markets and with different characteristics (Bitcoin, Gold, Nasdaq,

Table 8
Kendall correlation for GUB decomposition time series.

	Bitcoin	Gold	Brent	Nasdaq	SP500	Real Estate
			Good–Good			
Bitcoin	1	0.0492	−0.0217	0.0267	0.0133	0.0105
Gold	0.0492	1	0.0498	0.0917	0.1048	0.1124
Brent	−0.0217	0.0498	1	0.1482[**]	0.1754[**]	0.1052[**]
Nasdaq	0.0267	0.0917	0.1482[**]	1	0.826[**]	0.5207[**]
SP500	0.0133	0.1048	0.1754[**]	0.826[**]	1	0.62[**]
Real Estate	0.0105	0.1124	0.1052[**]	0.5207[**]	0.62[**]	1
			Usual–Usual			
Bitcoin	1	0.0214	0.0296[*]	0.0108	−0.008	−0.0169
Gold	0.0214	1	0.0552	−0.0328[*]	−0.04	0.0394[**]
Brent	0.0296[*]	0.0552	1	0.1101	0.1454[**]	0.0542[**]
Nasdaq	0.0108	−0.0328[*]	0.1101	1	0.6372[**]	0.2089[*]
SP500	−0.008	−0.04	0.1454[**]	0.6372[**]	1	0.2777[**]
Real Estate	−0.0169	0.0394[**]	0.0542[**]	0.2089[*]	0.2777[**]	1
			Bad–Bad			
Bitcoin	1	0.0459	0.0676	0.1191	0.1331	0.1104
Gold	0.0459	1	0.0758[*]	0.064	0.0831	0.1482[*]
Brent	0.0676	0.0758[*]	1	0.2501[*]	0.3058[**]	0.2327[*]
Nasdaq	0.1191	0.064	0.2501[*]	1	0.8764[**]	0.5092[**]
SP500	0.1331	0.0831	0.3058[**]	0.8764[**]	1	0.5971[**]
Real Estate	0.1104	0.1482[*]	0.2327[*]	0.5092[**]	0.5971[**]	1
			Good–Usual and Usual–Good			
Bitcoin	0	0.0171	−0.0218[*]	0.0221	0.0106	−0.0076
Gold	0.0141	0	0.003	−0.0226	−0.0139	0.0016
Brent	0.0239[*]	0.0223	0	0.0175	0.0269	0.0410
Nasdaq	0.0011	−0.0347	0.0559[**]	0	0.078	0.0892
SP500	0.0085	−0.0284	0.0677[**]	0.1072	0	0.1045
Real Estate	0.0051	0.0103	0.0119	0.0953	0.0691	0
			Good–Bad and Bad–Good			
Bitcoin	0	−0.0076	0.0129	−0.0099	−0.005	0.0137
Gold	0.003	0	−0.0313	−0.1017[**]	−0.1282[**]	−0.0654[**]
Brent	0.0065	0.0049	0	0.0113	0.0219	−0.0078
Nasdaq	0.023	−0.0573	0.0231	0	0.0835	0.0749
SP500	0.0189	−0.0712	0.0255	0.0875	0	0.0825
Real Estate	0.0248	−0.0735	−0.0034	−0.0728[**]	−0.0778[**]	0
			Usual–Bad and Bad–Usual			
Bitcoin	0	0.0291	0.0037	0.0366[*]	0.0364[**]	−0.0004
Gold	−0.0035	0	−0.0004	−0.0534	−0.0573	0.0153
Brent	0.0146	0.0297	0	0.1005[**]	0.1141	0.055[**]
Nasdaq	0.0112	0.0281	0.0791	0	0.1283	0.13[**]
SP500	0.0178	0.0179	0.0908[**]	0.1033	0	0.129[**]
Real Estate	−0.0126	0.0244	0.0156	0.1271	0.1379[**]	0

Note: [**] and [*] mean significant at 1% and 5%, respectively.

S&P-500, Brent and Dow Jones Real Estate), we compare the results obtained from the usual minimum KS distance and GUB method and find that the thresholds estimate using GUB method are lower volatile and more Gaussian. Besides, risk univariate estimate using GUB approach show lower excess of realized loss and lower overestimates. We conclude that GUB method shows better results than KS distance since it is a methodology that fits the central part of the distribution, while KS only fits the data from the tail of the distribution and, as a consequence, GUB uses a larger number of observations than KS distance.

In addition, we analyze the results for risk measurement of a portfolio composed of the six assets above using 3 multivariate methodologies: multivariate normal distribution, t-Student copula and multivariate dependence GUB decomposition. We find that our approach, without exceeding the excess loss limit (unlike the multivariate normal), is more economical in terms of capital consumption than the t-Student copula. Moreover, the GUB decomposition allows us to analyze the extreme dependence of asset returns without selecting a priori the starting percentile of the tail, as the extreme dependence co-movements measures do, and also allows us to estimate empirically the extreme dependence indexes, avoiding the implicit restrictions of the elliptic and Archimedean copulas. The results show that Gold is safe-have asset when the relationship is Good–Bad, whereas if the relationship is of the type Usual–Bad, the investors replace equities for Bitcoin, seeking potentially higher returns in trade-offs for higher risk-taking.

Finally, from the results obtained by applying the proposed methodology based on a low computational intensity partitioning of the series of individual asset returns, we provide answers to several relevant questions: estimating the threshold for the beginning of the tail of the distribution and analysis of the extreme dependence between the different components into which the original series of returns of the assets, included in the portfolio, are divided.

CRediT authorship contribution statement

Mariano González-Sánchez: Conceptualization, Methodology, Software, Data curation, Writing – original draft, Visualization, Investigation, Supervision, Software, Validation, Writing – review & editing. **Juan M. Nave Pineda:** Conceptualization, Methodology, Software, Data curation, Writing – original draft, Visualization, Investigation, Supervision, Software, Validation, Writing – review & editing.

Table 9
Dependence indexes for GUB decomposition time series.

	Bitcoin	Gold	Brent	Nasdaq	SP500	Real Estate
			Good–Good			
Bitcoin	0.1629[**]	0.0146	0.0072[**]	0.0186	0.0186[*]	0.0145[**]
Gold	0.0146	0.0605	0.0063	0.0076	0.009	0.0083
Brent	0.0072[**]	0.0063	0.0622[**]	0.0144[*]	0.0181[**]	0.0113[*]
Nasdaq	0.0186	0.0076	0.0144[*]	0.1052[*]	0.0763[*]	0.0405[*]
SP500	0.0186[*]	0.009	0.0181[**]	0.0763[*]	0.1155[**]	0.0509[**]
Real Estate	0.0145[**]	0.0083	0.0113[*]	0.0405[*]	0.0509[**]	0.0929[**]
			Usual–Usual			
Bitcoin	0.6973[**]	0.6166[**]	0.5953[**]	0.5579[**]	0.5501[**]	0.5586[**]
Gold	0.6166[**]	0.8718[**]	0.7524[**]	0.7037[**]	0.6994[**]	0.7108[**]
Brent	0.5953[**]	0.7524[**]	0.8539[**]	0.7002[**]	0.6943[**]	0.7023[**]
Nasdaq	0.5579[**]	0.7037[**]	0.7002[**]	0.8001[**]	0.7357[**]	0.688[**]
SP500	0.5501[**]	0.6994[**]	0.6943[**]	0.7357[**]	0.7882[**]	0.6911[**]
Real Estate	0.5586[**]	0.7108[**]	0.7023[**]	0.688[**]	0.6911[**]	0.8044[**]
			Bad–Bad			
Bitcoin	0.1248[**]	0.0077	0.0151[**]	0.0134[**]	0.0141[**]	0.0147[*]
Gold	0.0077	0.0632	0.0093	0.0059	0.0073	0.0105[*]
Brent	0.0151[**]	0.0093	0.077[**]	0.0193[**]	0.0227[**]	0.0171[*]
Nasdaq	0.0134[**]	0.0059	0.0193[**]	0.0942[**]	0.0709[**]	0.0394[**]
SP500	0.0141[**]	0.0073	0.0227[**]	0.0709[**]	0.0956[**]	0.0457[**]
Real Estate	0.0147[*]	0.0105[*]	0.0171[*]	0.0394[**]	0.0457[**]	0.1013[**]
			Good–Usual and Usual–Good			
Bitcoin		0.1374[**]	0.1435[**]	0.1282[**]	0.1276[**]	0.1335[**]
Gold	0.0352		0.047	0.0435	0.0399	0.0448
Brent	0.0473[**]	0.0524[**]		0.0422[**]	0.0398[**]	0.0438[**]
Nasdaq	0.0759	0.0895	0.0846		0.029	0.0626
SP500	0.084[**]	0.0961[*]	0.0906[**]	0.0391		0.063[*]
Real Estate	0.0687[*]	0.0771[*]	0.0732[*]	0.0509	0.0417	
			Good–Bad and Bad–Good			
Bitcoin		0.0091	0.0069	0.0095[*]	0.0116[*]	0.0075[*]
Gold	0.0109		0.0112[**]	0.0161[**]	0.0167[**]	0.0149[*]
Brent	0.0073[**]	0.0033		0.0056	0.0044	0.0071[**]
Nasdaq	0.0098	0.0081	0.005		0.0010	0.002
SP500	0.0116[**]	0.0101[*]	0.0059	0.0002		0.0015
Real Estate	0.0089	0.0073[*]	0.0078	0.0015	0.0002	
			Usual–Bad and Bad–Usual			
Bitcoin		0.0417	0.0498[**]	0.0631[*]	0.0626[**]	0.0697[**]
Gold	0.1074[**]		0.0608[**]	0.0781[*]	0.0757[**]	0.0828[**]
Brent	0.1016[**]	0.0504		0.0691[*]	0.0683[**]	0.0771[**]
Nasdaq	0.1017[**]	0.0492	0.0522[**]		0.0247	0.0599[**]
SP500	0.0992[**]	0.0458	0.0484[**]	0.0231		0.054[**]
Real Estate	0.1004[**]	0.0452	0.0521[**]	0.0534[*]	0.0497[**]	

Note: [**] and [*] mean significant at 1% and 5%, respectively.

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